وزارة التعليم العالي والبحث العلمي

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Properties of Discrete Fourier Transform (DFT)

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The DFT exhibits a number of useful properties and operational relationships that are similar to the properties of the continuous Fourier transform.

This Lecture investigates properties Discrete Fourier transform (DFT);

- The DFT as a Linear Transformation
- Periodicity, linearity, and symmetry properties
- Circular Symmetries of a Sequence.
- Multiplication of Two DFTs and Circular Convolution:

The DFT as a Linear Transformation

The formulas for the DFT and IDFT given above may be expressed as

$$X(K) = \sum_{N=0}^{N-1} x(n) W_N^{kn} \qquad k = 0,1,2,\dots\dots\dots N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0,1,2,\dots\dots\dots N-1$$

Where, by definition,

$$W_N = e^{-\frac{j2\pi}{N}}$$

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and (N-1) complex additions. Hence the N – point DFT values can be computed in a total of N^2 complex multiplications and N (N-1) complex additions.

Let us define an N – point vector x_N of the signal sequence *x(n), n* = 0, 1,…*N-1*, an N – point vector X_N of frequency samples, and an N × N matrix W_N

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$X_N = W_{NXN}$$

$$x_N = \frac{1}{N} W_N^* X_N$$

Where W_N^* denotes the complex conjugate of the matrix W_N

Example:-

Compute the DFT of the four – point sequence $x(n) = [0 \quad 1 \quad 2 \quad 3]$

Solution

The first step is to determine the matrix W_4. By exploiting the periodicity property of W_4 and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Properties of the DFT

In this section we present the important properties of the DFT. The notation used below to denote the N-point DFT pair *x(n)* and *X(k)* is

$$x(n) \overset{DFT}{\longleftrightarrow} X(k)$$

Periodicity, linearity, and symmetry properties:

Periodicity. If *x(n)* and *X(k)* are an N-point DFT pair, then

$$x(n+N)= x(n) \qquad \text{for all } n$$

$$X(k+N)= X(k) \qquad \text{for all } k$$

Linearity. If

$$x_1(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_1(k)$$

And

$$x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} X_2(k)$$

Then for any real-valued or complex-valued constants a_1 and a_2

$$a_1x_1(n) + a_2x_2(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k)$$

Circular Symmetries of a Sequence.

As we have seen, the *N*-point DFT of a finite duration sequence, *x(n)* of length $L \leq N$ is equivalent to the *N*- point DFT of a periodic sequence $x_p(n)$, of period *N*, which is obtained by periodically extending *x(n)*, that is

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

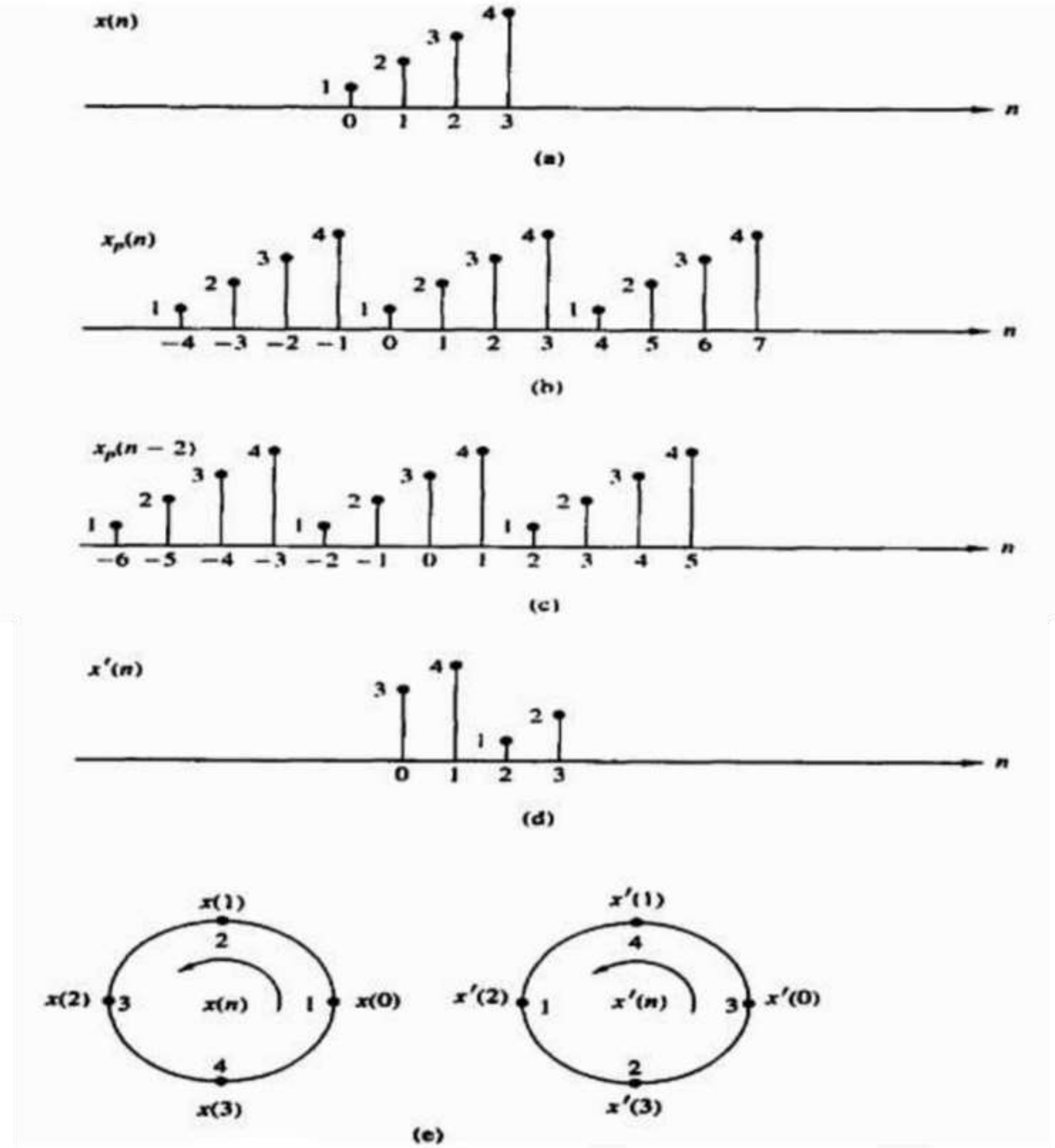
Now suppose that we shift the periodic sequence $x_p(n)$by k units to the right. Thus we obtain another periodic sequence

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

The infinite-duration sequence

$$x'(n) = \begin{cases} x'_p & 0 \le n \le N-1 \\ 0 & otherwise \end{cases}$$

Is related to the original sequence x(n) by a circular shift. This relationship is illustrated as shown in the figure below for *N*=4.



In general, the circular shift of the sequence can be represented as the index modulo- *N*. Thus we can write

$$x'(n) = x(n-k, modulo\ N)$$
$$= x\left((n-k)\right)_N$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x\left((n-2)\right)_4$$

Which implies that

$$x'(0) = x\left((-2)\right)_4 = x(2)$$

$$x'(1) = x\left((-1)\right)_4 = x(3)$$

$$x'(2) = x\left((0)\right)_4 = x(0)$$

$$x'(3) = x\left((1)\right)_4 = x(1)$$

Thus we conclude that a circular shift of an *N*-point sequence is equivalent to a linear shift of its periodic extension, and vice versa.

Multiplication of Two DFTs and Circular Convolution

Suppose that we have two finite – duration sequences of length *N*, $x_1(n)$ and $x_2(n)$. Their respective *N*-point DFTs are

$$X_1(k) = \left\{\begin{matrix} N-1 \\ n=0 \end{matrix}\right. \quad x_1(n)e^{-j2\pi nk/N} \qquad k = 0,1,2,\ldots..N-1$$

$$X_2(k) = \left\{\begin{matrix} N-1 \\ n=0 \end{matrix}\right. \quad x_2(n)e^{-j2\pi nk/N} \qquad k = 0,1,2,\ldots..N-1$$

If we multiply the two DFTs together, the result is a DFT, say $X_3(k)$, of a sequence $x_3(n)$ of length *N*. Let us determine the relationship between $x_3(n)$ and the sequence $x_1(n)$ and $x_2(n)$. *We* have

$$X_3(k) = X_1(k)X_2(k) \qquad k=0,1,2,\ldots..,N-1$$

The IDFT of ($X_3(k)$) is

$$X_3(m) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{j2\pi mk/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) e^{j2\pi km/N}$$

The inner sum in the brackets has the form

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a = 1 \\ \dfrac{1 - a^N}{1 - A} & a \neq 1 \end{cases}$$

Where ***a*** is defined as

$$a = e^{j2\pi(m-n-1)/N}$$

We observe that $a = 1$ when $m\text{-}n\text{-}l$ is a multiple of N. On the other hand, $a^N = 1$ for any value of $a \neq 1$ Consequently

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & l = m - n + pN = ((M-N))N, p \text{ an interger} \\ 0 & \text{otherwise} \end{cases}$$

Then the desired expression for $x_3(m)$ in the form

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \qquad m = 0,1,,2,\ldots.N-1$$

The expression above has the form of a convolution sum. However, it is not the ordinary linear convolution. Instead, the convolution sum involves the index $((m\text{-}n))_N$ and is called *circular convolution*. Thus we conclude that multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in the time domain.

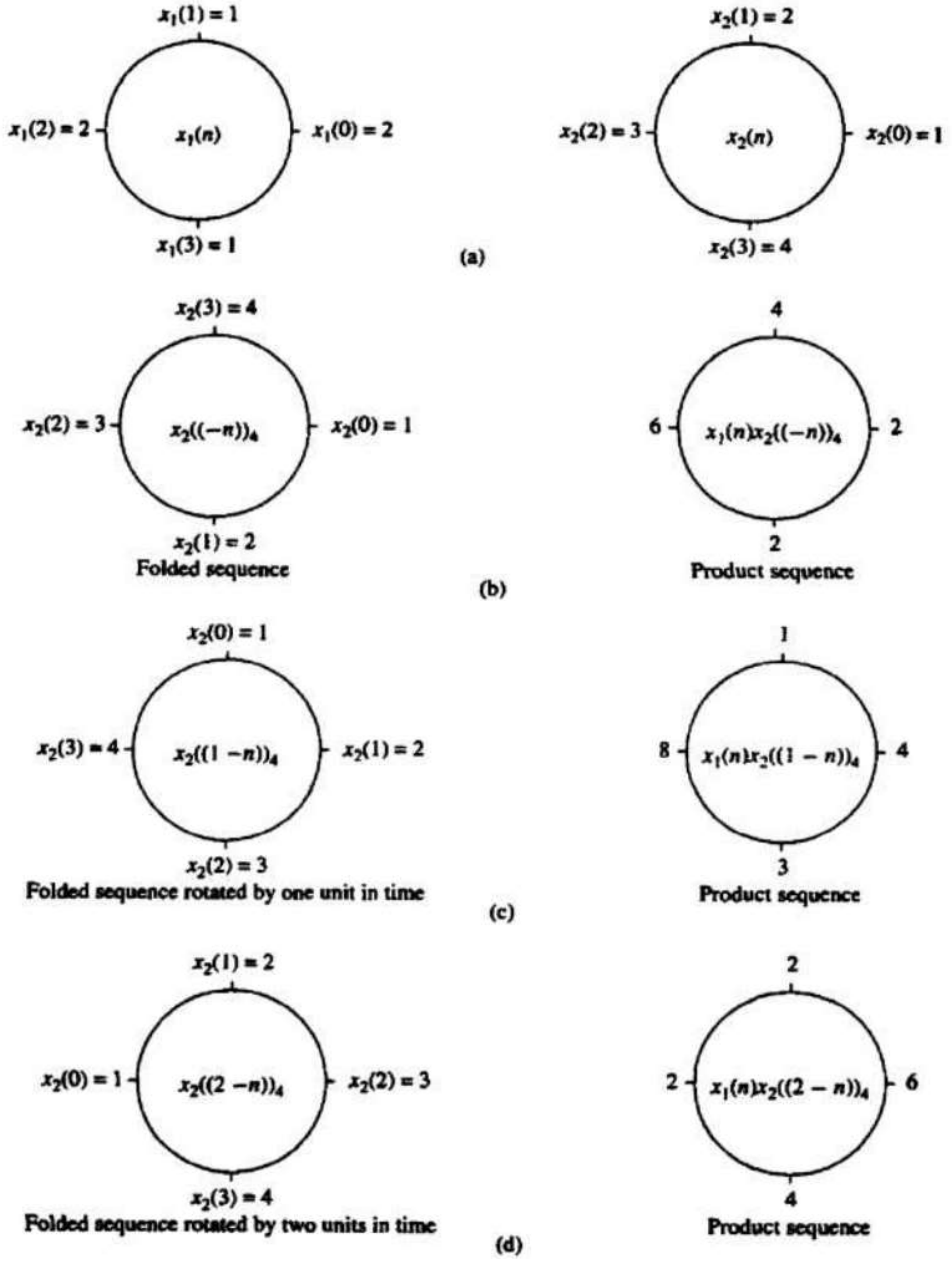
Example:-

Perform the circular convolution of the following two sequences:

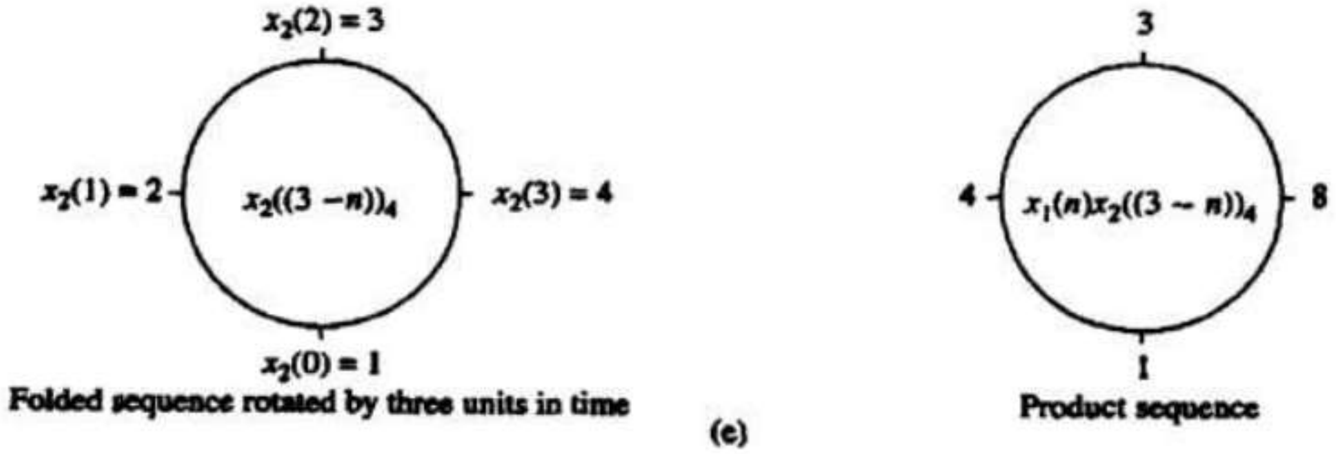
$$x_1(n) = \{2, \underset{\uparrow}{1}, 2, 1\}$$

$$x_2(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$$

Solution:



x1(1) = 1
x1(2) = 2
x1(n)
x1(0) = 2
x1(3) = 1
(a)
x2(1) = 2
x2(2) = 3
x2(n)
x2(0) = 1
x2(3) = 4
x2(3) = 4
x2(2) = 3
x2((−n))4
x2(0) = 1
x2(1) = 2
Folded sequence
(b)
4
6
x1(n)x2((−n))4
2
2
Product sequence
x2(0) = 1
x2(3) = 4
x2((1 −n))4
x2(1) = 2
x2(2) = 3
Folded sequence rotated by one unit in time
(c)
1
8
x1(n)x2((1 − n))4
4
3
Product sequence
x2(1) = 2
x2(0) = 1
x2((2 −n))4
x2(2) = 3
x2(3) = 4
Folded sequence rotated by two units in time
(d)
2
2
x1(n)x2((2 − n))4
6
4
Product sequence



$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

Example:

By means of the DFT and IDFT, determine the sequence $x_3(n)$ corresponding to the circular convolution of the sequences $x_1(n)$ and $x_2(n)$ given in previous example.

Solution:

First we compute the DFTs of $x_1(n)$ and $x_2(n)$. The four – point DFT of $x_1(n)$ is

$$X_1(k) = \sum_{n=0}^{3} x_1(n)\, e^{-\frac{j2\pi nk}{4}} \qquad k = 0,1,,2,3$$
$$= 2 + e^{-\frac{j\pi k}{2}} + 2\, e^{-j\pi k} + e^{-\frac{j3\pi k}{2}}$$

Thus

$X_1(0)=6$ $X_1(1)=0$ $X_1(2)=2$ $X_1(3)=0$

The DFT of $x_2(n)$ is

$$X_2(k) = \sum_{n=0}^{3} x_2(n)\, e^{-\frac{j2\pi nk}{4}} \qquad k = 0,1,,2,3$$

$$= 1 + 2e^{-\frac{j\pi k}{2}} + 3\, e^{-j\pi k} + 4e^{-\frac{j3\pi k}{2}}$$

Thus

$X_2(0)=10$ $X_2(1)=-2+2j$ $X_2(2)=-2$ $X_2(3)=-2-2j$

When we multiply the two DFTs, we obtain the product

$$X_3(k) = X_1(k)\, X_2(k)$$

Or, equivalently

$X_3(0)=60$ $X_3(1)=0$ $X_3(2)=-4$ $X_3(3)=0$

Now, the IDFT of $X_3(k)$ is

$$x_3\ (n) = \sum_{n=0}^{3} X_3(k)\, e^{\frac{j2\pi nk}{4}} \qquad\qquad n = 0,1,,2,3$$

$$= \frac{1}{4}(60 - 4e^{j\pi n}\ )$$

Thus

$x_3(0)=14$ $\quad$ $x_3(1)=16$ $\quad$ $x_3(2)=14$ $\quad$ $x_3(3)=16$

Which is the result obtained in previous example.

We conclude this section by formally stating this important property of the DFT.

Circular convolution.

If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n) \textcircled{N} x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)X_2(k)$$