



وزارة التعليم العالي والبحث العلمي

جامعة ديالى

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Analysis of Discrete –Linear Time Invariant

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This Lecture covered the impulse response of discrete time systems and the response to arbitrary input signals using digital convolution.

The major topics discussed in this unit are included in the following outline.

- Format of Difference Equation
- System Representation Using Its Impulse Response
- Digital Convolution

Difference Equations and Impulse Responses

Format of Difference Equation

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1y(n - 1) + \dots + a_Ny(n - N) \quad (1)$$

$$= b_0x(n) + b_1x(n - 1) + \dots + b_Mx(n - M) \quad (2)$$

Or

$$y(n) = - \sum_{i=1}^N a_i y(n - i) + \sum_{j=0}^M b_j x(n - j) \quad (3)$$

Notice that $y(n)$ is the current output, which depends on the past output samples $y(n - 1), \dots, y(n - N)$, the current input sample $x(n)$, and the past input samples, $x(n - 1), \dots, x(n - M)$.

We will examine the specific difference equations in the following examples.

Example:-

Given the following difference equation:

$$y(n) = 0.25y(n - 1) + x(n)$$

Identify the nonzero system coefficients.

Solution:

Comparison with Equation (2) leads to

$$b_0 = 1$$

$$-a_1 = 0.25$$

System Representation Using Its Impulse Response:

A linear time-invariant system can be completely described by its unit- impulse response, which is defined as the system response due to the impulse input $\delta(n)$ with zero initial conditions, depicted in Figure (4).

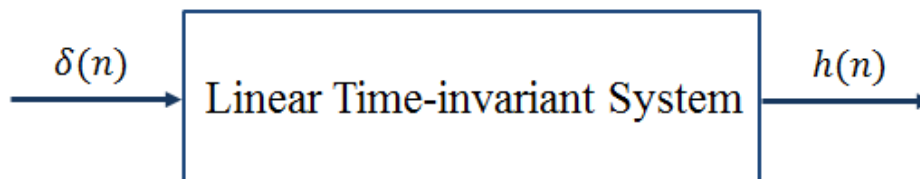


Figure (4): Unit-impulse response of the linear time-invariant system.

With the obtained unit-impulse response $h(n)$, we can represent the linear time-invariant system in Figure (5).

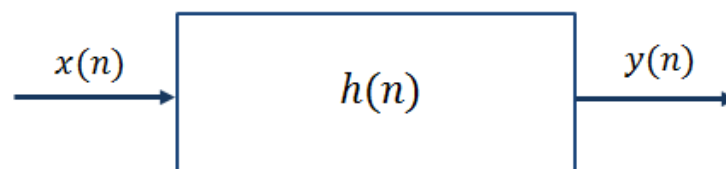


Figure (5): Representation of a linear time-invariant system using the impulse response.

Example:

Given the linear time-invariant system

$$y(n) = 0.5x(n) + 0.25x(n - 1) \text{ with an initial condition } x(-1) = 0,$$

- a) Determine the unit-impulse response $h(n)$.
- b) Draw the system block diagram.
- c) Write the output using the obtained impulse response.

Solution:

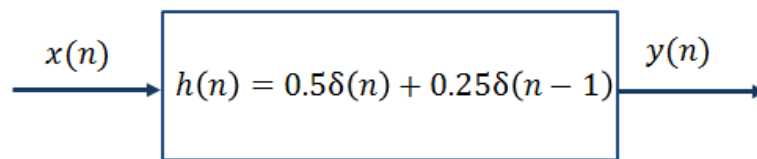
- a) According to Figure 4, let $x(n) = \delta(n)$, then

$$h(n)=y(n)=0.5x(n)+0.25x(n-1)=0.50\delta(n)+0.25\delta(n-1)$$

Thus, for this particular linear system, we have

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & \text{elsewhere} \end{cases}$$

- b) The block diagram of the linear time-invariant system is shown as



- c) The system output can be rewritten as

$$y(n) = h(0)x(n) + h(1)x(n - 1).$$

In general, we can express the output sequence of a linear time-invariant system from its impulse response and inputs as

$$y(n) = \dots + h(-1)x(n + 1) + h(0)x(n) + h(1)x(n - 1) + h(2)x(n - 2) + \dots \quad (3)$$

Equation (3) is called the *digital convolution sum*, which will be explored in a later section.

We can verify Equation (3) by substituting the impulse sequence $x(n) = \delta(n)$ to get the impulse response

$$h(n) = \dots + h(-1)\delta(n + 1) + h(0)\delta(n) + h(1)\delta(n - 1) + h(2)\delta(n - 2) + \dots,$$

Example: Given the difference equation

$$y(n) = 0.25y(n - 1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

- Determine the unit-impulse response $h(n)$.
- Draw the system block diagram.
- Write the output using the obtained impulse response.
- For a step input $x(n) = u(n)$, verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Equation 3).

Solution:

a) Let $x(n) = \delta(n)$, then

$$h(n) = 0.25h(n - 1) + \delta(n).$$

To solve for $h(n)$, we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$

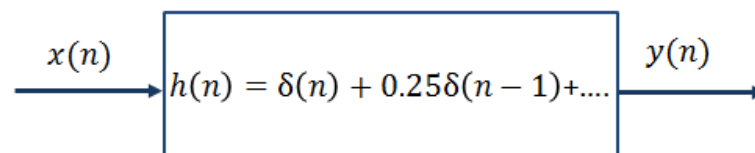
$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$

$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.25 + 0 = 0.0625$$

With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n - 1) + 0.0625\delta(n - 2) + \dots$$

b) The system block diagram is given in Figure below.



c) The output sequence is a sum of infinite terms expressed as

$$\begin{aligned} y(n) &= h(0)x(n) + h(1)x(n - 1) + h(2)x(n - 2) + \dots \\ &= x(n) + 0.25x(n - 1) + 0.0625x(n - 2) + \dots \end{aligned}$$

d) From the difference equation and using the zero-initial condition, we have

$$\begin{aligned}
 y(n) &= 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0 \\
 n = 0, y(0) &= 0.25y(-1) + x(0) = u(0) = 1 \\
 n = 1, y(1) &= 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25 \\
 n = 2, y(2) &= 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125 \\
 &\dots
 \end{aligned}$$

Applying the convolution sum in Equation (3) yield

$$\begin{aligned}
 y(n) &= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \dots \\
 n = 0, y(0) &= x(0) + 0.25x(-1) + 0.0625x(-2) + \dots \\
 &= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \dots = 1 \\
 n = 1, y(1) &= x(1) + 0.25x(0) + 0.0625x(-1) + \dots \\
 &= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \dots = 1.25 \\
 n = 2, y(2) &= x(2) + 0.25x(1) + 0.0625x(0) + \dots \\
 &= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \dots = 1.3125
 \end{aligned}$$

Digital Convolution:

Convolution is a mathematical operation on two functions producing a third function that is affected at any time by all previous input values. Convolution is the most important and fundamental concept in signal processing and analysis. By using convolution, we can construct the output of any LTI system for any arbitrary input signal, if we know the impulse response of that system. The impulse response goes by a different name in some applications. If the system being considered is a filter, the impulse response is called the *filter kernel*, the convolution kernel, or simply, the *kernel*.

In image processing, the impulse response is called the point spread function. The convolution is performed by sliding the kernel along the input signal.

Given a linear time-invariant system, we can determine its unit-impulse response $h(n)$, which relates the system input and output. To find the output sequence $y(n)$ for any input sequence

$x(n)$, we write the *digital convolution* as shown in Equation (3) as

$$\begin{aligned}
 y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
 &= \dots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \dots \\
 y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
 &= \dots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \dots
 \end{aligned} \tag{4}$$

Using a conventional notation, we express the digital convolution

$$y(n) = h(n) * x(n). \tag{5}$$

Note that for a causal system, which implies its impulse response

$$h(n) = 0 \text{ for } n < 0,$$

The lower limit of the convolution sum begins at 0 instead of 1, that is

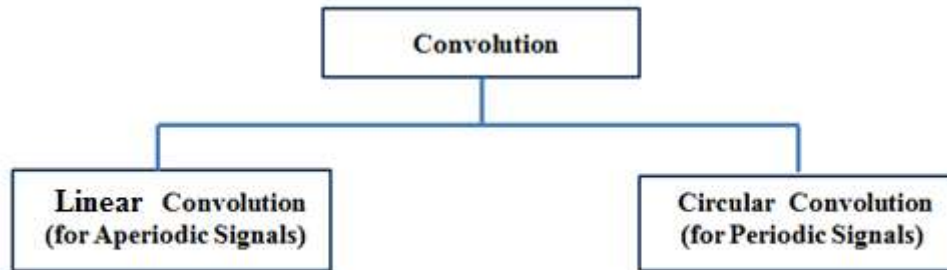
$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} x(k)h(n-k).$$

We will focus on evaluating the *convolution sum* based on Equation (4). Let us examine first a few outputs from Equation (4):

$$\begin{aligned}
 y(0) &= \sum_{k=-\infty}^{\infty} x(k)h(-k) = \dots + x(-1)h(1) + x(0)h(0) + x(1)h(-1) + x(2)h(-2) + \dots \\
 y(1) &= \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \dots + x(-1)h(2) + x(0)h(1) + x(1)h(0) + x(2)h(-1) + \dots \\
 y(2) &= \sum_{k=-\infty}^{\infty} x(k)h(2-k) = \dots + x(-1)h(3) + x(0)h(2) + x(1)h(1) + x(2)h(0) + \dots \\
 &\dots
 \end{aligned}$$

Classification of convolution:

Convolution can be classified into two categories according to the signals that will be convolved as shown in the figure below.



Graphical Method:-

The steps involved in using the graphical approach are as follows:

1. Plot both sequences, $x(k)$ and $h(k)$, as functions of k .
2. Choose one of the sequences, say $h(k)$, and time-reverse it to form the sequence $h(-k)$.
3. Shift the time-reversed sequence by n .

[if $n > 0$, this corresponds to a shift to the right (delay), whereas if $n < 0$, this corresponds to a shift to the left (advance)].

4. Multiply the two sequences $x(k)$ and $h(n - k)$ and sum the product for all values of k .

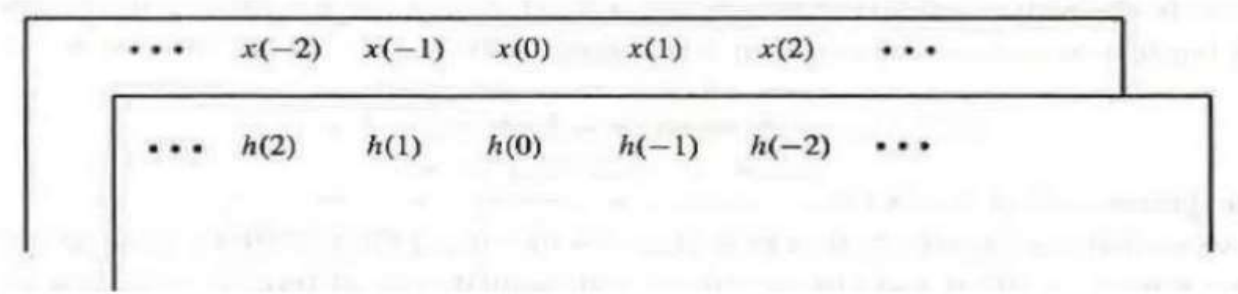
The resulting value will be equal to $y(k)$. This process is repeated for all possible shifts, n .

Tabular Method (Slide Rule Method):-

The slide rule method is particularly convenient when both $x(k)$ and $h(n)$ are finite in length and short in duration. The steps involved in the slide rule method are as follows:

Write the values of $x(k)$ along the top of a piece of paper, and the values of $h(-k)$ along the top of another piece of paper as illustrated in figure. Line up the two sequences values $x(0)$ and $h(0)$, multiply each pair of numbers, and add the

products to form the values of $y(0)$.



Example:-

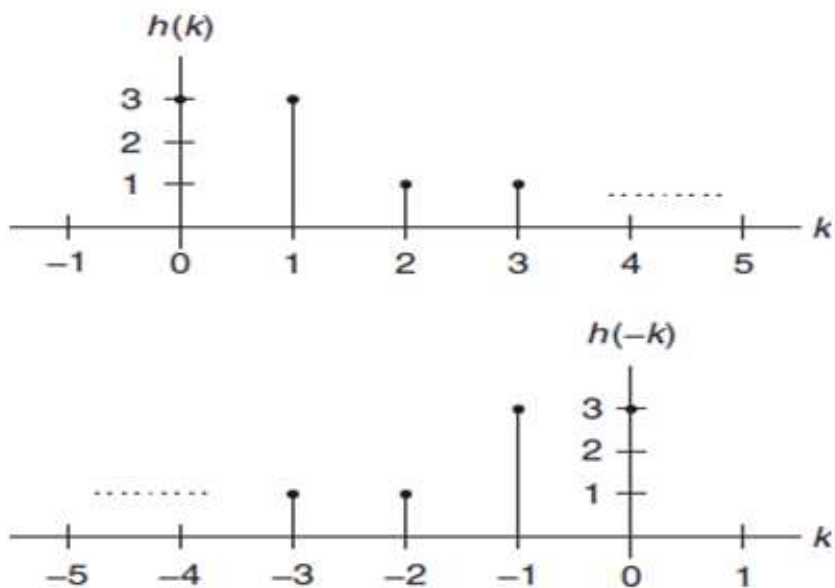
Given a sequence,

$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & \text{elsewhere} \end{cases}$$

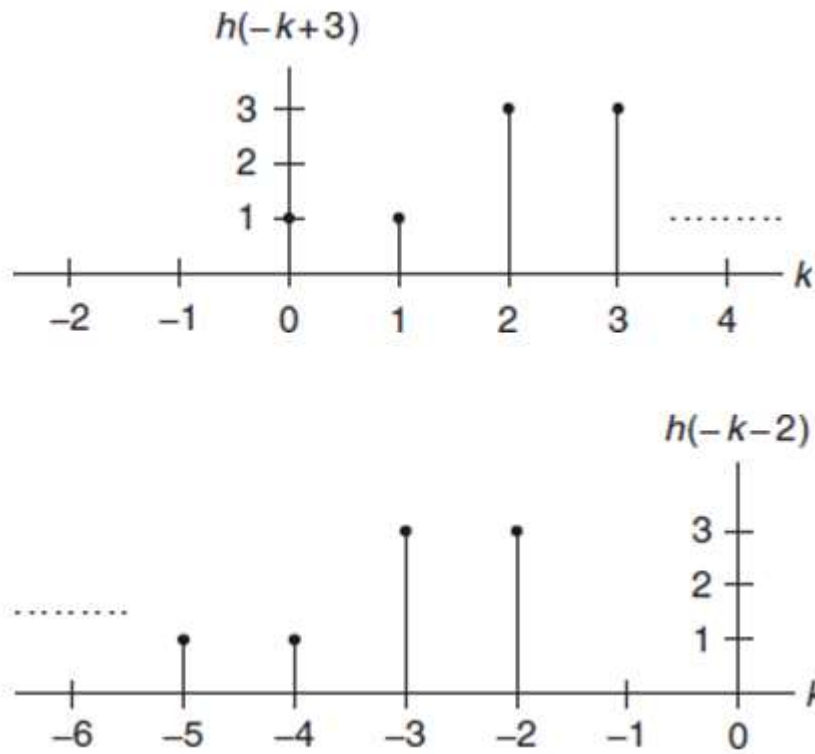
Where k is the time index or sample number,

- a) Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.
- b) Sketch the shifted sequences $h(-k + 3)$ and $h(-k - 2)$.

Solution: a)

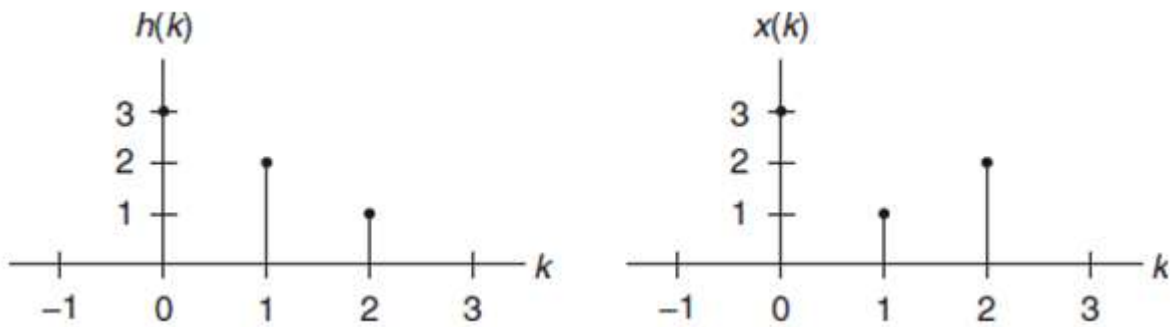


b)



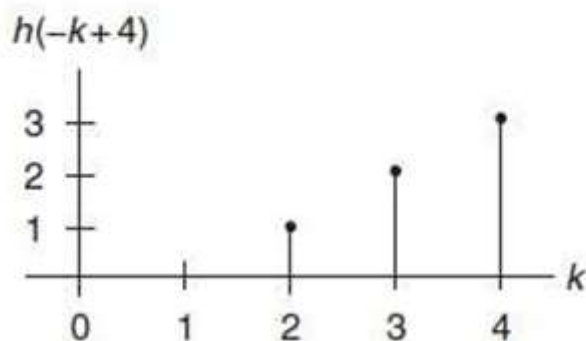
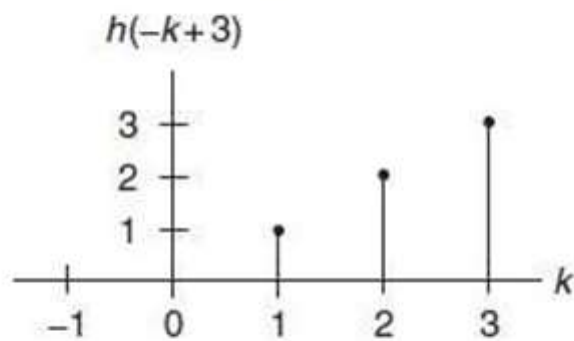
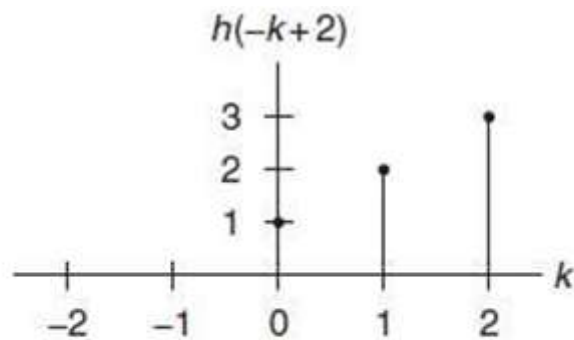
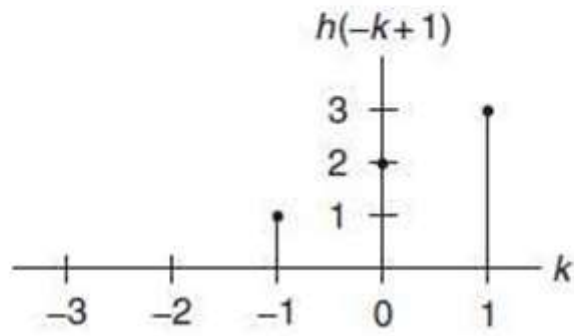
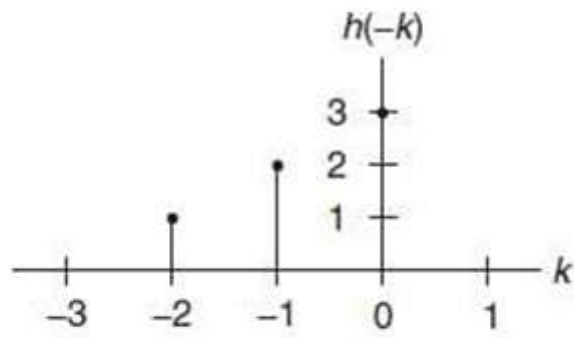
Example:

Using the following sequences defined in the Figure below, evaluate the digital convolution



- a) By the graphical method.
- b) By applying the formula directly.

Solution: a)



We can compute the convolution sum as

sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$

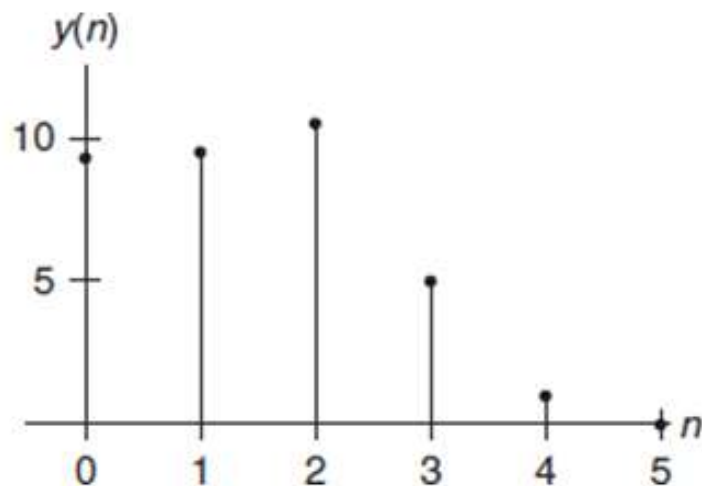
sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$

sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$

sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$

sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$

sum of product of $x(k)$ and $h(5-k)$: $y(n) = 0$ for $n > 4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.



b) Applying Equation (4) with zero initial conditions leads to

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$$n = 0, y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$$

$$n = 1, y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$$

$$n = 2, y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$$

$$n = 3, y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$$

$$n = 4, y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$$

$$n \geq 5, y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

Example:

Given the following two rectangular sequences

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & \text{otherwise} \end{cases} \text{ and } h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

Convolve them using the table method

Solution:

Digital convolution steps via the table.

-
- Step 1. List the index k covering a sufficient range.
 - Step 2. List the input $x(k)$.
 - Step 3. Obtain the reversed sequence $h(-k)$, and align the rightmost element of $h(n-k)$ to the leftmost element of $x(k)$.
 - Step 4. Cross-multiply and sum the nonzero overlap terms to produce $y(n)$.
 - Step 5. Slide $h(n-k)$ to the right by one position.
 - Step 6. Repeat step 4; stop if all the output values are zero or if required.
-

Convolution sum using the table method.

$k:$	-2	-1	0	1	2	3	4	5	
$x(k):$			3	1	2				
$h(-k):$	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)
