وزارة التعليم العالي والبحث العلمي

جامعة ديالى

كلية العلوم

قسم علوم الحاسبات

Discrete Time Signals (DTS)

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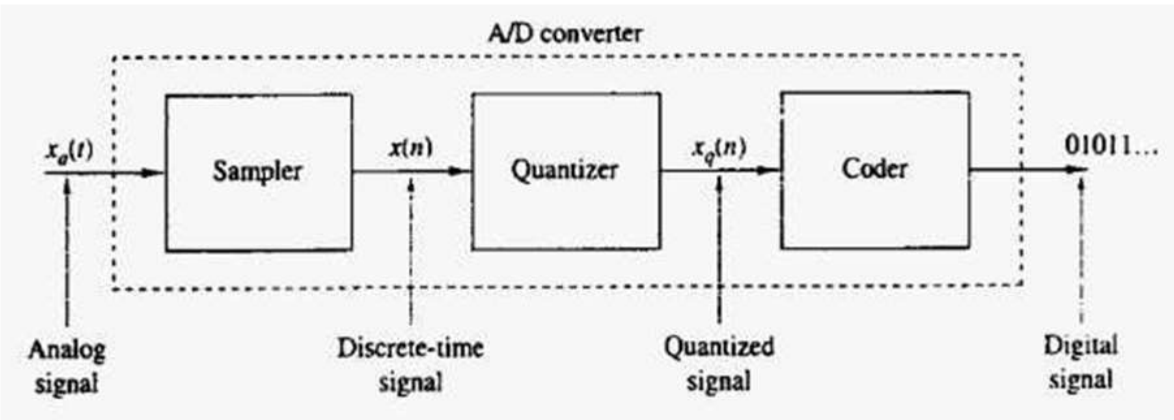
The major topics discussed in this Lecture unit are included in the following outline.

- Analog – to – digital (ADC) and Digital – to Analog Conversion (DAC),
- Sampling of Analog Signals,
- The sampling theorem,
- Quantization of Continuous – Amplitude signal,

Analog to Digital (ADC) and Digital to Analog Conversion(DAC):

Most signals of practical interest, such as biological signals are analog. To process analog signals by digital means, it is first necessary to convert them in digital form. That is, to convert them to a sequence of numbers having finite precision. This procedure is called analog – to – digital (A/D) conversion.

Conceptually, we view A/D conversion as a three – step process. This process is illustrated in the figure below:



1- Sampling. This is the conversion of continuous – time signal into a discrete – time signal obtained by taking "samples" of the continuous – time signal at discrete – time instants. Thus, if *xa(t)* is the input to the sampler, the output is *xa(nT)* ≡ *x(n)*, where *T* is called the *sampling interval*.

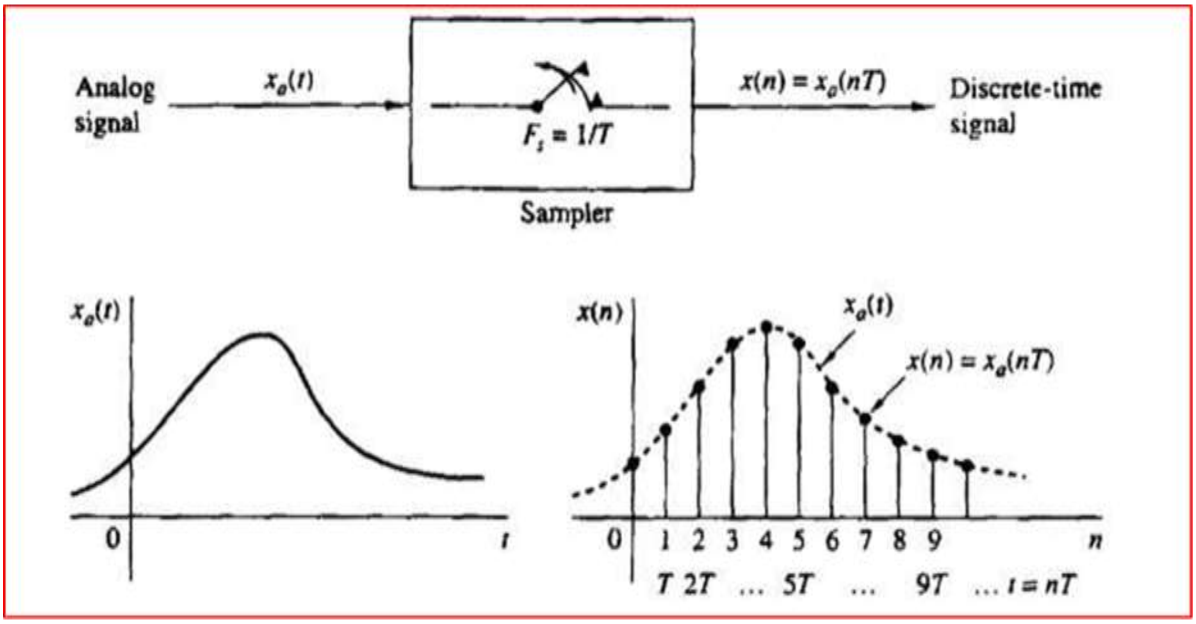
2- Quantization. This is the conversion of a discrete – time continuous – valued signal into a discrete – time, discrete – valued (digital) signal. The value of each signal sample is represented by a value selected from a finite set of possible values. The difference between the unquantized sample *x(n)* and the quantized output *xq(n)* is called the *quantization error*.

3- Coding. In the coding process, each discrete value *xq(n)* is represented by *a* b-bit binary sequence.

Sampling of Analog Signals:

There are many ways to sample an analog signal. We limit our discussion to periodic or uniform sampling, which is the type of sampling used most often in practice. This is described by the relation:

$$x(n) = xa(nT), \qquad -\infty < n < \infty$$



The time interval *T* between successive samples is called the *sampling period* or *sample interval* and its reciprocal $1/T = F_s$ is called the *sampling rate* or *sampling frequency*.

The variables t and n are linearly related through the sampling period T or, equivalently, through the sampling rate $F_s = 1/T$, as

$$t = nT = \frac{n}{F_s}$$

As a consequence of above equation, there exists a relationship between the frequency variable F (or Ω) for analog signals and the frequency variable f (or ω) for discrete – time signals. To establish this relationship, consider an analog sinusoidal signal of the form:

$$x_a(t) = A\cos(2\pi Ft + \theta)$$

$$xa(nT) \equiv x(n) = A\cos(2\pi FnT + \theta)$$

$$x_a(nT) \equiv x(n) = A\cos\left(\frac{2\pi Fn}{F_s}T + \theta\right)$$

The frequency variables F and f are linearly related as:

$$f = \frac{F}{F_s}$$

Or, equivalently, as $\omega = \Omega$

The frequency variable f is relative or normalized frequency. We can use f to determine the frequency F in hertz only if the sampling frequency F_s is known.

The relations are summarized in following table:

Continuous-time signals		**Discrete - time signals**	
$\Omega = 2\pi F$		$\omega = 2\pi f$	
radians/ sec	*Hz*	*radians/ sample*	*cycles/ sample*

$\omega = \Omega T,\ f = F/F_s$ →

← $\Omega = \omega/T,\ F = f.F_s$

Continuous-time signals	Discrete - time signals
$-\infty < \Omega < \infty$	$-\pi/T \le \omega \le \pi/T$
$-\infty < F < \infty$	$-F_s/2 \le f \le F_s/2$

From these relations we observe that

$$f_{max} = \frac{F_s}{2} = \frac{1}{2T}$$

$$\omega_{max} = \pi F_s = \frac{\pi}{T}$$

The sampling theorem:

If the highest frequency contained in an analog signal *xa(t)* is $F_{max} = B$ and the signal is sampled at a rate $F_s > 2F_{max} \equiv 2B$, then *xa(t)* can be exactly recovered from its sample values using the interpolation function:

$$g(t) = \frac{sin2\pi Bt}{2\pi Bt}$$

Thus *xa(t)* may be expressed as:

$$x_a(t) = \sum_{n=-\infty}^{\infty} xa\left(\frac{n}{F_s}\right) g\left(t - \frac{n}{F_s}\right)$$

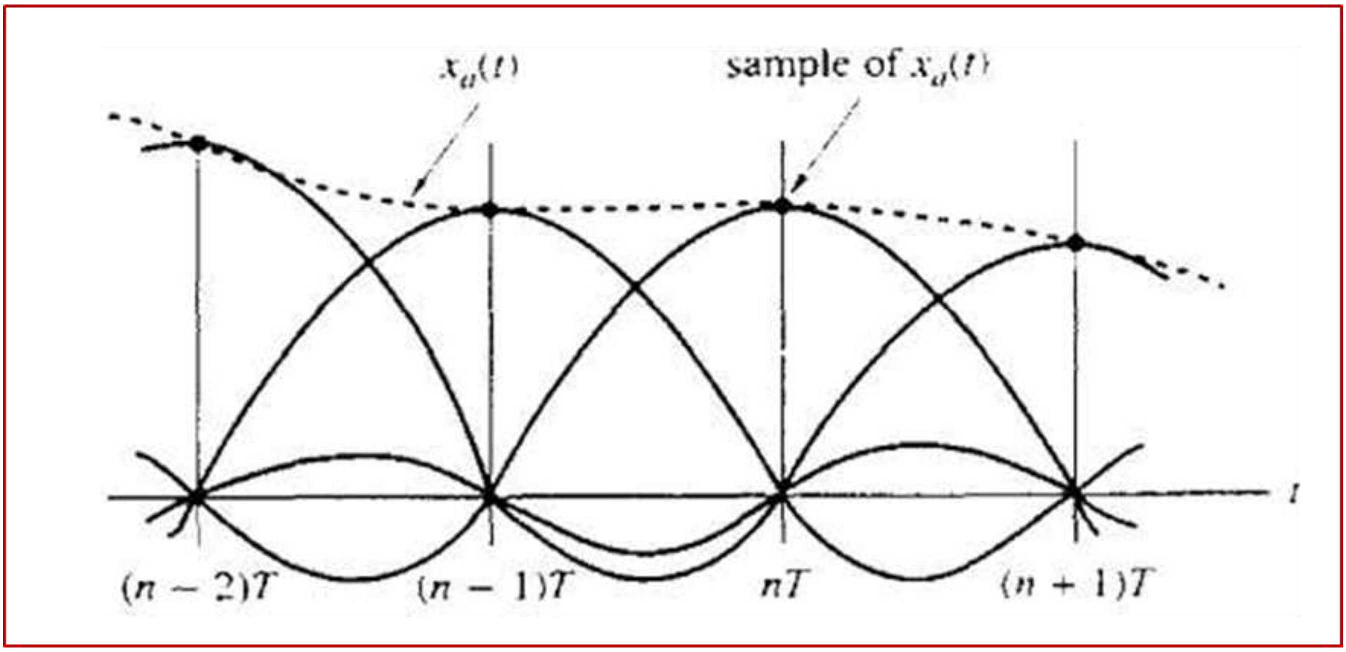
Where $x_a(n/F_s) = x_a(nT) \equiv x(n)$ are the samples of $x_a(t)$.

When the sampling of $x_a(t)$ is performed at the minimum sampling rate $F_s = 2B$, the reconstruction formula becomes:

$$x_a(t) = \sum_{n=-\infty}^{\infty} xa\left(\frac{n}{2B}\right) \frac{sin2\pi B(t - \frac{n}{2B})}{2\pi B(t - \frac{n}{2B})}$$

And the sampling rate $F_N = 2B$ is called *Nyquist rate*.

Figure below illustrate the ideal D/A conversion process using the interpolation function



Example:-

Consider the analog signal

$$x_a(t) = 3cos50\pi t + 10sin300\pi t - cos100\pi t$$

What is the Nyquist rate for this signal?

Solution:-

The frequencies present in the signal above are

$F_1 = 25$ Hz, $F_2 = 150$ Hz, $F_3 = 50$ Hz

Thus $F_{max} = 150$ Hz and $F_s > 2F_{max} = 300$ Hz $= F_N$

Example:

Consider the analog signal

$$x_a = 3cos2000\pi t + 5sin6000\pi t + 10cos12000\pi t$$

a) What is the Nyquist rate for this signal?

b) Assume now that we sample this signal using a sampling rate $F_s = 5000$ samples/sec. what is the discrete – time signal obtained after sampling?

c) What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation?

Solution:

a) The frequencies existing in the analog signal are

$F_1 = 1KHz$, $F_2 = 3$ KHz, $F_3 = 6$ KHz

Thus $F_{max} = 6$ KHz, and according to the sampling theorem

$F_s > 2\ F_{max} = 12\ KHz$ and the Nyquist rate is $F_N = 12\ KHz$

b) Since we have chosen $F_s = 5\ KHz$, the folding frequency is $F_s / 2 = 2.5\ KHz$

And this is the maximum frequency that can be represented uniquely by the sampled signal. We obtain

$$x(n) = x_a(nT) = x_a\left(\frac{n}{F_S}\right)$$

$$= 3cos2\pi\left(\frac{1}{5}\right)n + 5sin2\pi\left(\frac{3}{5}\right)n + 10cos2\pi\left(\frac{6}{5}\right)n$$

$$= 3cos2\pi\left(\frac{1}{5}\right)n + 5sin2\pi\left(1-\frac{2}{5}\right)n + 10cos2\pi\left(1+\frac{1}{5}\right)n$$

$$= 3cos2\pi\left(\frac{1}{5}\right)n + 5sin2\pi\left(-\frac{2}{5}\right)n + 10cos2\pi\left(\frac{1}{5}\right)n$$

Finally, we obtain

$$x(n) = 13cos2\pi\left(\frac{1}{5}\right)n - 5sin2\pi\left(\frac{2}{5}\right)n$$

c) Since only the frequency components at 1 *KHz* and 2 *KHz* are present in the sampled signal, the analog signal we can recover is

$$y_a(t) = 13cos2000\pi t - 5sin4000\pi t$$

Which is obviously different from the original signal $x_a(t)$. This distortion of the original analog signal was caused by the *aliasing effect*, due to the low sampling rate used.

Quantization of Continuous – Amplitude Signal:

The process of converting a discrete – time continuous – amplitude signal into a digital signal by expressing each sample value as a finite (instead of an infinite) number of digits, is called *quantization*. The error introduced in representing the continuous – valued signal by a finite set of discrete value levels is called *quantization error* or *quantization noise*.

We denote the *quantizer* operation on the samples $x(n)$ as $Q[x(n)]$ and let $x_q(n)$ denote the sequence of *quantized samples* at the output of the *quantizer*. Hence.

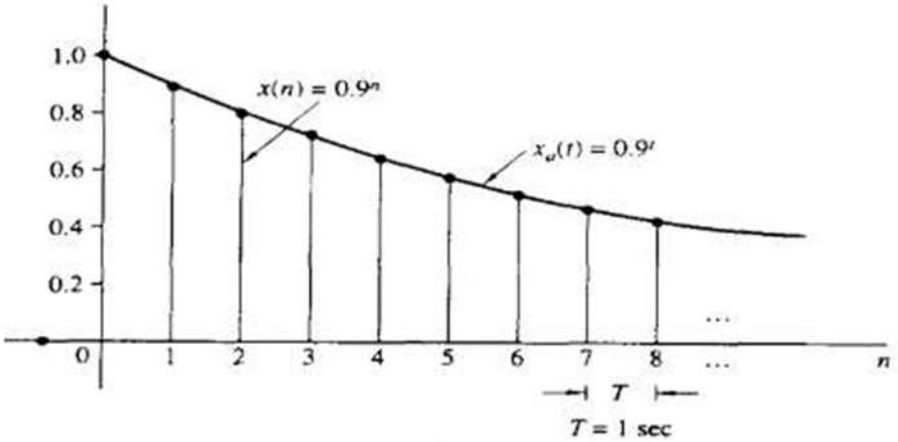
$$x_q(n) = Q[x(n)]$$

Then the *quantization error* is a sequence $e_q(n)$ defined as the difference between the *quantized* value and the *actual sample* value. Thus

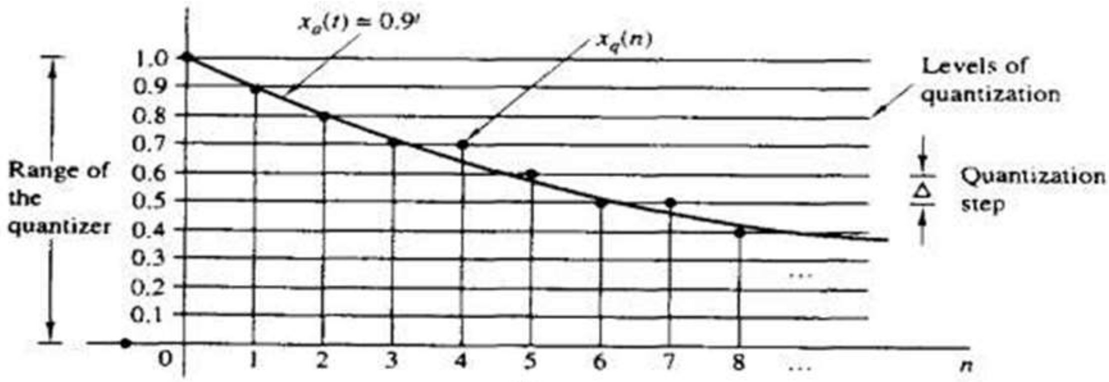
$$e_q(n) = x_q(n) - x(n)$$

Let us consider the discrete – time signal

$$x(n) = \begin{cases} 0.9^n, & n \geq 0 \\ 0 & n < 0 \end{cases}$$



(a)



(b)

TABLE 1.2 NUMERICAL ILLUSTRATION OF QUANTIZATION WITH ONE SIGNIFICANT DIGIT USING TRUNCATION OR ROUNDING

n	$x(n)$ Discrete-time signal	$x_q(n)$ (Truncation)	$x_q(n)$ (Rounding)	$e_q(n) = x_q(n) - x(n)$ (Rounding)
0	1	1.0	1.0	0.0
1	0.9	0.9	0.9	0.0
2	0.81	0.8	0.8	−0.01
3	0.729	0.7	0.7	−0.029
4	0.6561	0.6	0.7	0.0439
5	0.59049	0.5	0.6	0.00951
6	0.531441	0.5	0.5	−0.031441
7	0.4782969	0.4	0.5	0.0217031
8	0.43046721	0.4	0.4	−0.03046721
9	0.387420489	0.3	0.4	0.012579511