Simple Harmonic Oscillator

(3-1) Introduction:

When a particle oscillates about its means position along a straight line under the action of a force which (i) is directed toward the mean position and (ii) is proportional to the displacement at any instant from this position, the motion of the particle is said to be simple harmonic, and the oscillating particle is called a simple harmonic oscillator, or a linear harmonic oscillator.

The harmonic oscillator is an important example of periodic motion because it serves (يفيد) as an exact or an approximate model for many problems in classical or quantum physics. At temperatures above $0°K$, the atoms in a crystal are temporarily (مؤقتاً) displaced from their normal positions in the structure, due to absorption of thermal energy. consequently, interatomic (تداخلات) forces (قوى) obeying Hooke's Law act on the displaced atoms. Under the action of such restorting (ارجاعية) forces each atom vibrates about its normal position which is the correct position in the ideal structure. Thus the vibrations of each atom are similar to those of a simple harmonic oscillator.

(3-2) one-Dimensional simple harmonic oscillator in Classical Mechanics.

Let the equilibrium position of a linear harmonic oscillator of mass (m) be the origin o, and the straight line of its motion be the x-axis. When x is its displacement from o, the restoring force F is given by;

$$F = -kx$$

where k is a positive constant.

According to the second law of motion;

$$F = m\frac{d^2x}{dt^2}$$

Therefore, its Newtonian equation of motion is;

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x = -\omega^2 x ,$$

where $\omega = \sqrt{\frac{k}{m}}$

The general solution of this equation is;

$$x = A\sin(\omega t + \theta) \qquad \text{———— (1)}$$

where A and θ are constants, which may be determined from the initial conditions. Eq (1) shows that the particle performs simple harmonic oscillations with frequency

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad (2)$$

where ω is called the angular frequency of oscillations

The Potential energy $V(x)$ at the displacement x is given by;

$$V = \int_0^V dv = \int_0^x \frac{dv}{dx}\, dx$$

But
$$\frac{dv}{dx} = -F = -(-kx)$$
$$= kx$$

$$\therefore \quad V = \int_0^x kx\, dx = \frac{1}{2}kx^2$$

$$= \frac{1}{2}m\omega^2 x^2 \qquad (3)$$

$$K = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}mA^2\omega^2\cos^2(\omega t+\theta)$$

$$= \frac{1}{2}m\omega^2 A^2\left[1-\sin^2(\omega t+\theta)\right]$$

$$= \frac{1}{2}m\omega^2(A^2 - x^2) \qquad (4)$$

Therefore, according to classical mechanics the total energy E of the oscillator is given by;

$$E = K + V$$

$$= \frac{1}{2}m\omega^2(A^2 - x^2) + \frac{1}{2}m\omega^2 x^2$$

$$\boxed{E = \frac{1}{2}m\omega^2 A^2} \qquad (5)$$

Eq (3) shows that the potential energy has a parabolic form. It means that the particle moves in a potential well of parabolic form. Eq (5) shows that the total energy of a particle oscillating with a constant frequency $\nu = \frac{\omega}{2\pi}$ is proportional to the square of the amplitude.

According to classical mechanics the particle can oscillate with any amplitude, consequently the total energy increases continuously with increase in the amplitude.

(3-3) One-Dimensional simple harmonic oscillator in Quantum mechanics.

(3-3-1) Wave equation for the oscillator

The time-independent Schrodinger wave equation for linear motion of a particle along the x-axis is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

or

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\psi = 0 \qquad (6)$$

where E is the total energy of the particle, V the potential energy, and ψ the wave-function for the particle, which is a function of x alone.

For a linear oscillator along the x-axis with the angular frequency w under a restoring force proportional to the displacement x, the potential energy is given by;

$$V = \frac{1}{2} m w^2 x^2 \qquad (7)$$

Substituting the value of V in eq(6) we get;

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2} m w^2 x^2\right)\psi = 0 \qquad (8)$$

or

$$\frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2} - \frac{m^2 w^2}{\hbar^2} x^2\right)\psi = 0 \qquad (9)$$

This is the Schrodinger wave equation for the oscillator.

تبسيط

(3-3-2) Simplification of the wave equation

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To simplify eq(9), we introduce a dimensionless

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independent variable y which is related to x by the equation;

$$y = ax \qquad (10)$$

So that

$$x = \frac{y}{a}$$

where

$$a = \sqrt{\frac{mw}{\hbar}}$$

Now we here $\frac{d\psi}{dx} = \frac{d\psi}{dy}\frac{dy}{dx} = a\frac{d\psi}{dy}$

and $$\frac{d^2\varphi}{dx^2} = \frac{d^2\varphi}{dy^2}\,\frac{dy}{dx}\,a = a^2\,\frac{d^2\varphi}{dy^2} \qquad \text{———} (11)$$

Substituting the Values of $\frac{d^2\varphi}{dx^2}$ and x^2 in eq(9), we get

$$a^2\frac{d^2\varphi}{dy^2} + \left[\frac{2mE}{\hbar^2} - a^4\frac{y^2}{a^2}\right]\varphi = 0$$

Dividing through by a^2

$$\frac{d^2\varphi}{d\cancel{x}^2} + \left[\frac{2mE}{a^2\hbar^2} - y^2\right]\varphi = 0$$

or $$\frac{d^2\varphi}{dy^2} + \left(\frac{2E}{\hbar\omega} - y^2\right]\varphi = 0 \qquad \text{———} (12)$$

or $$\frac{d^2\varphi}{dy^2} + (\lambda - y^2)\varphi = 0, \text{where } \lambda = \frac{2E}{\hbar\omega} \qquad \text{———} (13)$$

ولو أن

Though eq(12) is in a simplified form, it is not easy to solve it. Therefore, its solution in detail is not given here.

In the following paragraphs, we discuss the results of its solution.

(3-3-3) Eigen-Values of the total energy E_n

The wave equation for the Oscillator is satisfied only for discrete Values of total energies given by

$$\frac{2E}{\hbar\omega} = (2n+1)$$

or $E_n = \frac{1}{2}(2n+1)\hbar\omega$

$$E_n = (n + \tfrac{1}{2})\hbar\omega \quad \text{———} (14)$$

substituting $\hbar = \frac{h}{2\pi}$ and $\omega = 2\pi\nu$, this expression has the form;

$$E_n = (n + \tfrac{1}{2})h\nu \quad \text{———} (15)$$

where, $n = 0, 1, 2, \text{——}$, ω is the angular frequency and ν is the frequency of the classical harmonic oscillator, given by:

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

From eq(14), we get the following conclusions;

1- The lowest energy of the oscillator is obtained by putting $n = 0$ in eq(14) and it is;

$$E_0 = \frac{1}{2}\hbar\omega \quad \text{———} (16)$$

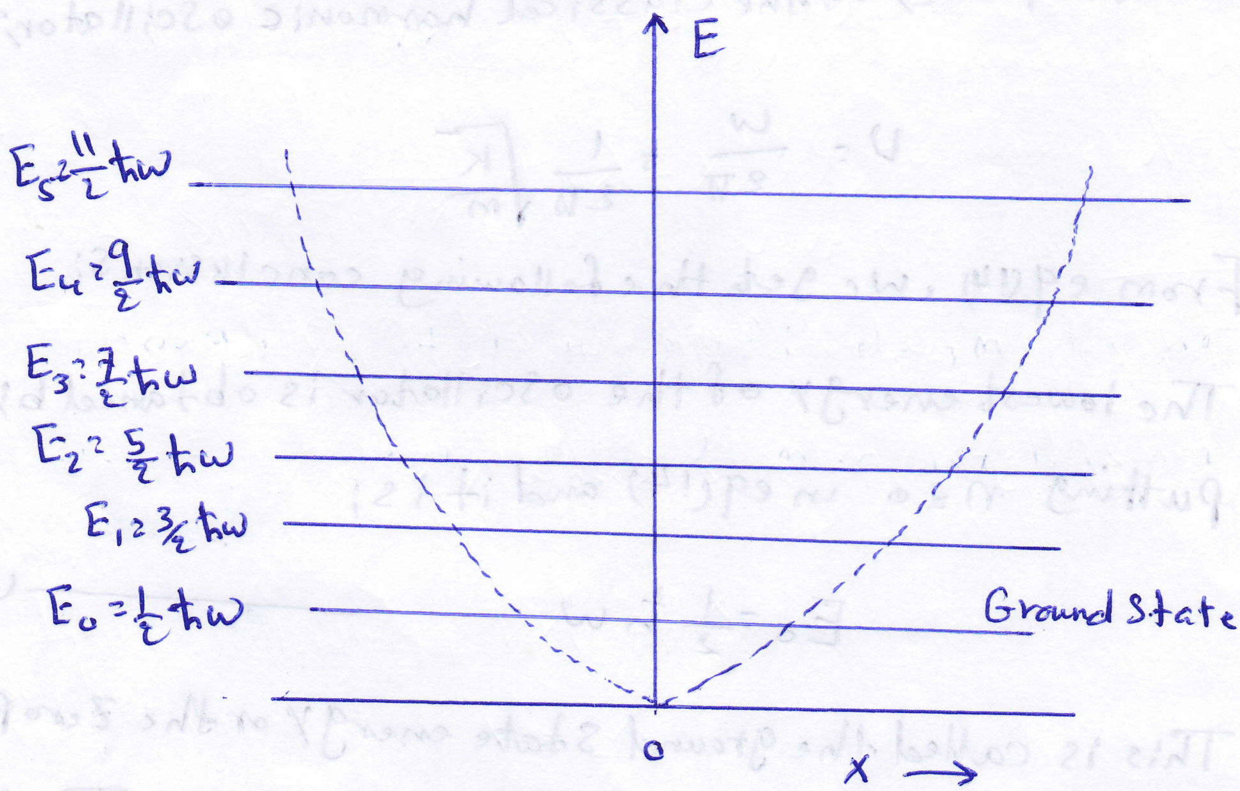
This is called the ground state energy or the zero point vibrational energy of the harmonic oscillator. The zero point energy is the characteristic result of quantum mechanics. The values of E_n in terms of E_0 are given by

$$E_n = (2n+1)E_0 \quad \text{———} (17)$$

where $n = 0, 1, 2, 3, \text{——}$

2- The eigen-values of the total energy depend only on one quantum number n. Therefore all the energy-levels of the oscillator are non-degenerate.

3- The متعاقبة successive energy-levels are equally spaced; the separation between two مجاور أو مجاورة adjacent energy-levels being $\hbar\omega$. The energy-level diagram for the harmonic oscillator is shown in figure below.



In the figure horizontal lines show the energy levels and the dashed curve is parabola representing the potential energy $V = \frac{1}{2}Kx^2$

We desire to obtain solutions $\psi(y)$ which satisfy equation (13) throughout the region $-\infty$ to $+\infty$ for y, and which are acceptable functions i.e, each function must be continuous, single valued and finite throughout the region.

Let us consider asymptotic solution i.e. $y \to \infty$. When y is Large we can neglect λ in comparison with y. Then equation (13) become:

$$\frac{d^2\psi}{dy^2} - y^2\psi = 0$$

The general solution of the above equation will be of the form.

$$\psi_{(y)} = A e^{-y^2/2} + B e^{y^2/2}$$

As $y \to \infty$, $e^{-y^2/2} \to 0$ and $e^{y^2/2} \to \infty$. Thus acceptable soultion will be.

$$\psi_{(y)} = A e^{-y^2/2}$$

We proceed to obtain the solution of wave equation (13) throughout configration space $(-\infty < y < +\infty)$, based upon the asymptotic soultion. Let us assume that the solution is of the form.

$$\psi_{(y)} = A H(y) e^{-y^2/2} \qquad (18)$$

where A is constant, and $H(y)$ is some function of y from equation (18)

$$\frac{d\psi}{dy} = AH'(y)e^{-y^2/2} - yAH(y)e^{-y^2/2}$$

and $$\frac{d^2\psi}{dy^2} = AH''(y)e^{-y^2/2} - yAH'(y)e^{-y^2/2} - AH(y)e^{-y^2/2} - yAH'(y)e^{-y^2/2} + y^2AH(y)e^{-y^2/2}$$

$$\therefore \frac{d^2\psi}{dy^2} = AH''(y)e^{-y^2/2} - 2yAH'(y)e^{-y^2/2} - AH(y)e^{-y^2/2} + y^2AH(y)e^{-y^2/2}$$

using $\frac{d^2\psi}{dy^2}$ and $\psi(y)$ in equation (13), we get

$$AH''(y)e^{-y^2/2} - 2yAH'(y)e^{-y^2/2} - AH(y)e^{-y^2/2} + y^2AH(y)e^{-y^2/2} + (\lambda - y^2)AH(y)e^{-y^2/2} = 0$$

$$\therefore Ae^{-y^2/2}[H''(y) - 2yH'(y) + (\lambda - 1)H(y) = 0 \quad \text{——} (19)$$

As $e^{-y^2/2}$ is an arbitrary function, we get

$$H''(y) - 2yH'(y) + (\lambda - 1)H(y) = 0$$

or

$$H'' - 2yH' + (\lambda - 1)H = 0 \quad \text{——} (20)$$

where primes denote differentiation with respect to y

Equation (20) is the Hermite differential equation for (H) in the variable "y". We try to obtain the solution of this equation in the form of frobenius power Serres.

بحل المعادلة (20) بهذه الطريقة (أي الطريقة متسلسلة فروبنيوس) ينتج الآتي

$$2n+1-\lambda=0$$

$$\lambda=2n+1$$

من تعريف المعادلة (13) $\lambda=\frac{2E}{\hbar\omega}$

$$\frac{2E}{\hbar\omega}=2n+1$$

$$2E=(2n+1)\hbar\omega \quad \div 2$$

$$\boxed{E=\left(n+\tfrac{1}{2}\right)\hbar\omega \quad \text{or} \quad E=\left(n+\tfrac{1}{2}\right)h\nu}$$

وهذه هي معادلة الاشتقاق. وتقدم شرحها في البند السابق.

أن معادلة هيرمايت التفاضلية من الدرجة n (حسب المعادلة 20) لها الصيغة الرياضية الآتية

$$H_n''(y)-2y\,H_n'(y)+2n\,H_n(y)=0 \qquad (21)$$

وحل هذه المعادلة يدعى متعددة حدود هيرمايت (Hermite Polynomal)

والصيغة الرياضية لهذا الحل هو:

$$\boxed{H_n(y)=(-1)^n\, e^{+y^2}\frac{d^n}{dy^n}e^{-y^2}} \qquad (22)$$

حيث أن $n=0,1,2,3$ ---

(3-4) Normalization of wave function:

The wave function for one-dimensional harmonic oscillator is given as:

$$\psi_{(y)} = A H(y) e^{-y^2/2}$$

where A is constant called normalization constant.

Since

$$y = \alpha x$$

$$\therefore \quad \psi_{(x)} = A H(\alpha x) e^{-\alpha^2 x^2/2}$$

Since energy depends on the integer n, the wave function corresponding to different energy levels E_n is expressed as

$$\psi_n(y) = A_n H_n(y) e^{-y^2/2} \qquad \text{——— (23)}$$

or

$$\psi_n(x) = A_n H_n(\alpha x) e^{-\alpha^2 x^2/2} \qquad \text{——— (24)}$$

We determine A_n, using normalization condition.

$$\int_{-\infty}^{+\infty} \psi \psi^* dx = 1$$

i.e

$$\int_{-\infty}^{+\infty} |\psi_{(x)}|^2 dx = 1$$

$$\therefore \int_{-\infty}^{\infty} |A_n|^2 H_n^2(\alpha x)\, e^{-\alpha^2 x^2}\, dx = 1$$

$$\therefore |A_n|^2 \int_{-\infty}^{+\infty} H_n^2(\alpha x)\, e^{-\alpha^2 x^2}\, dx = 1$$

we have $\displaystyle\int_{-\infty}^{\infty} H_n^2(\alpha x)\, e^{-\alpha^2 x^2}\, dx = \frac{1}{\alpha}\int_{-\infty}^{\infty} H_n^2(y)\, e^{-y^2}\, dy$

But the integral $\displaystyle\int_{-\infty}^{\infty} H_n^2(y)\, e^{-y^2}\, dy = \sqrt{\pi}\, 2^n\, n!$

$$|A_n|^2 \frac{\sqrt{\pi}\, 2^n n!}{\alpha} = 1$$

This gives $\quad A_n = \left(\dfrac{\alpha}{\sqrt{\pi}\, 2^n\, n!}\right)^{1/2}$

Therefore the normalized wave function for the harmonic oscillator is

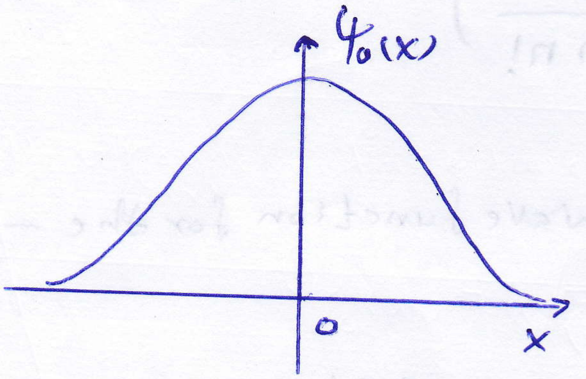
$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n\, n!}\right)^{\frac{1}{2}} H_n(\alpha x)\, e^{-\alpha^2 x^2/2} \quad \text{———} (25)$$

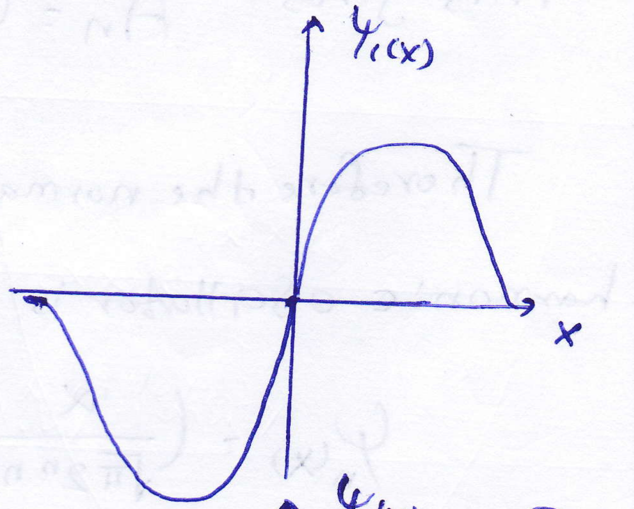
where $n = 0, 1, 2, 3, 4, \text{———}$

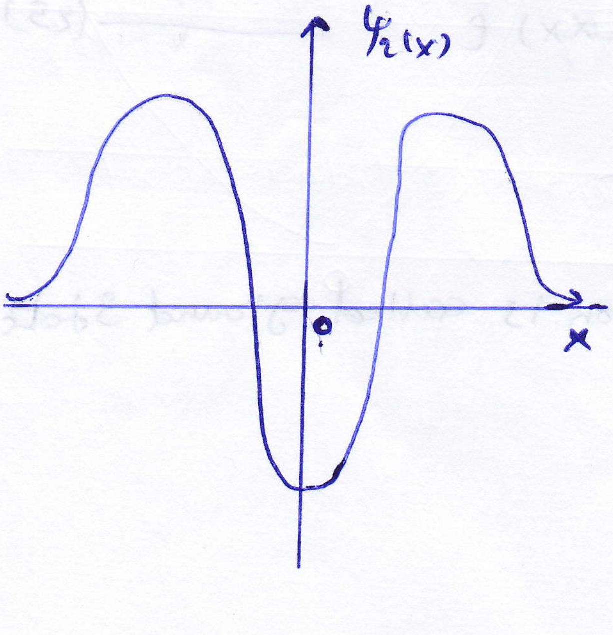
For $n = 0$, the wave function is called ground state wave function.

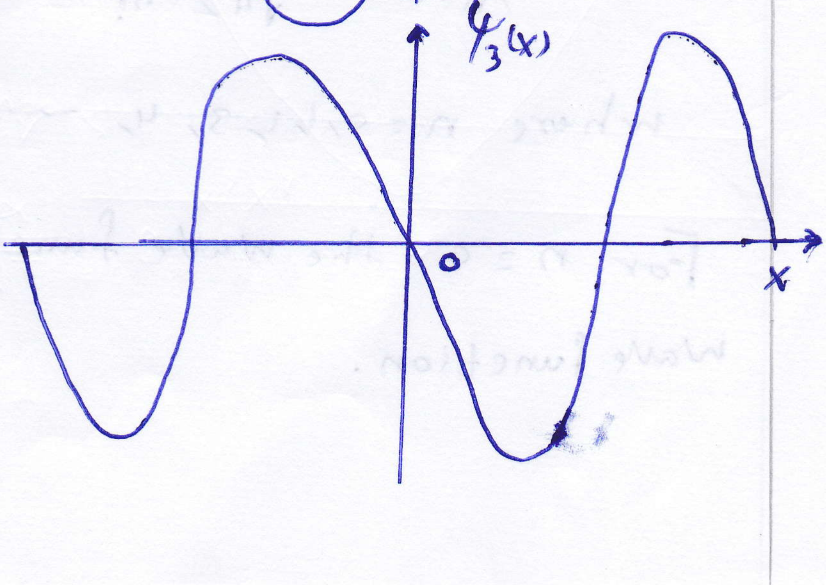
The first six Hermite Polynomials are given in the following table:

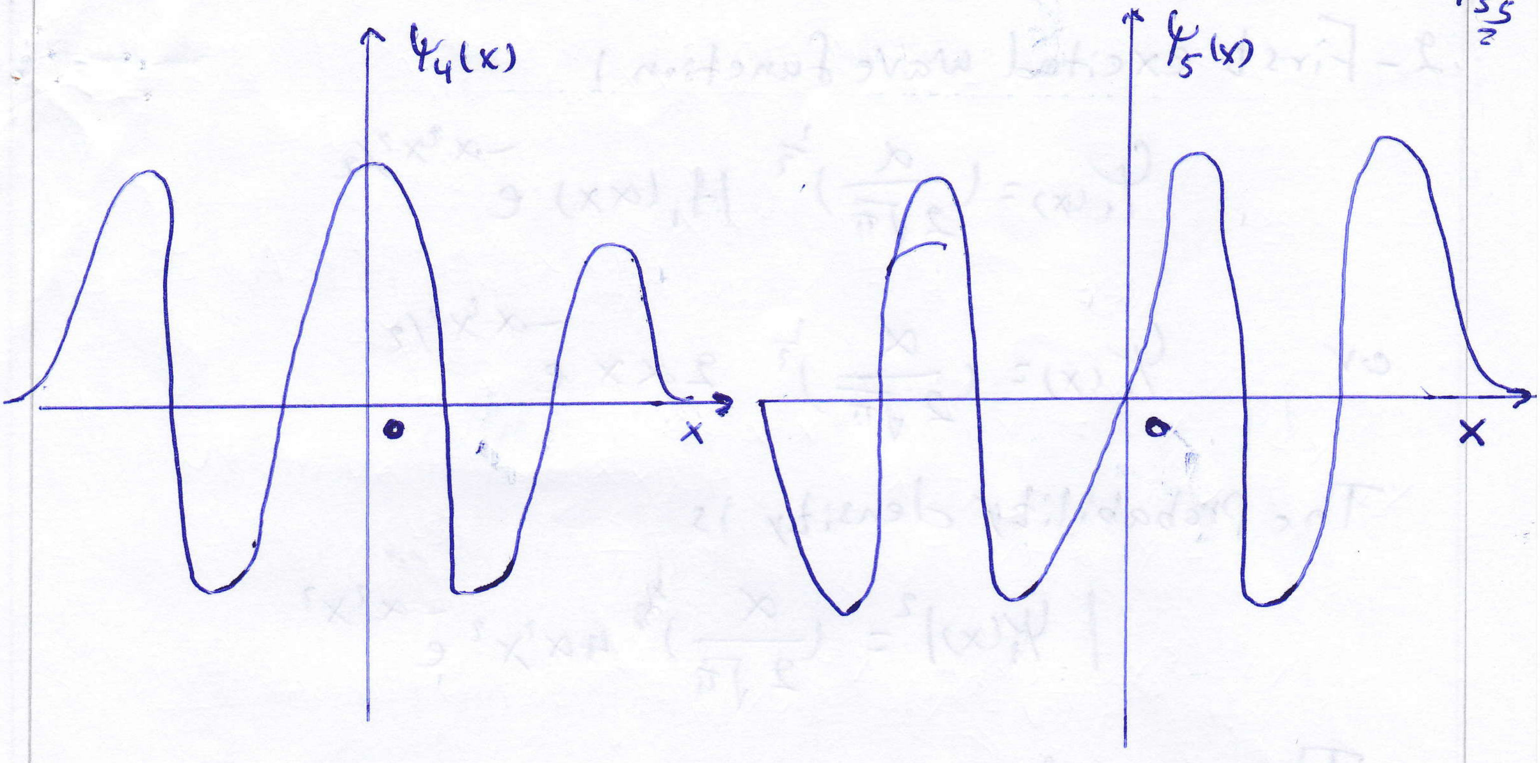
n	$\lambda = 2n+1$	E_n	$H_n(y)$
0	1	$\frac{1}{2}\hbar\omega$	$H_0(y) = 1$
1	3	$\frac{3}{2}\hbar\omega$	$H_1(y) = 2y$
2	5	$\frac{5}{2}\hbar\omega$	$H_2(y) = 4y^2 - 1$
3	7	$\frac{7}{2}\hbar\omega$	$H_3(y) = 8y^3 - 12y$
4	9	$\frac{9}{2}\hbar\omega$	$H_4(y) = 16y^4 - 48y^2 + 12$
5	11	$\frac{11}{2}\hbar\omega$	$H_5(y) = 32y^5 - 160y^3 + 120y$











The first six wave functions are shown in above.

The wave functions for different states are given as follows;

(1) Ground state function;

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} H_0(\alpha x)\, e^{-\alpha^2 x^2/2}$$

or

$$\psi_0(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha^2 x^2}$$

The probability density is

$$|\psi_0(x)|^2 = \left(\frac{\alpha}{\sqrt{\pi}}\right) e^{-\alpha^2 x^2}$$

2- First excited wave function :

$$\psi_1(x) = \left(\frac{\alpha}{2\sqrt{\pi}}\right)^{\frac{1}{2}} H_1(\alpha x)\, e^{-\alpha^2 x^2/2}$$

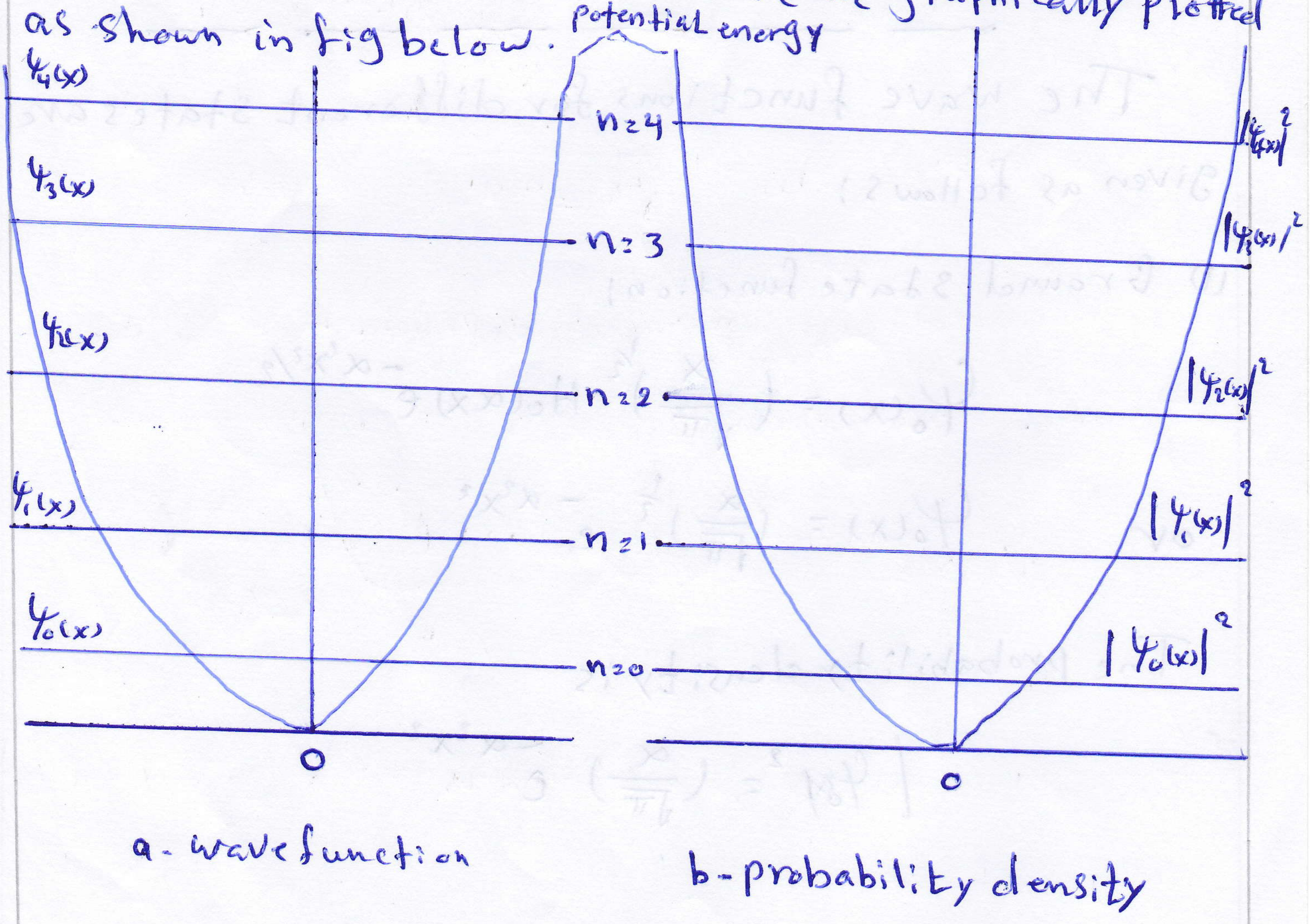
or

$$\psi_1(x) = \left(\frac{\alpha}{2\sqrt{\pi}}\right)^{\frac{1}{2}} 2\alpha x\, e^{-\alpha^2 x^2/2}$$

The probability density is

$$|\psi_1(x)|^2 = \left(\frac{\alpha}{2\sqrt{\pi}}\right) 4\alpha^2 x^2 e^{-\alpha^2 x^2}$$

The wave function and probability density for ground state and first four excited state are graphically plotted as shown in fig below.



a- wave function b- probability density

(3-5) Generating Function;

يمكن التعبير عن الدالة المولدة ب $g(s,y)$ التي تعرف بالشكل الآتي :-

$$g(s,y) = e^{y^2-(s-y)^2} = e^{-s^2+2ys} = \sum_{n=0}^{\infty} \frac{H_n(y)\, s^n}{n!} \qquad (26)$$

أي أن $H_n(y)$ تساوي $n!$ مضروباً في معامل S^n كما مرّ أن معكوس الدالة هو

$$\sum_n a_n s^n$$

فيما يلي نشتق المعادلة التفاضلية التي تحققها كثيرة حدود هيرمايت من الدرجة n باشتقاقه بالدالة المولدة $g(s,y)$ (لنأخذ المشتقة الجزئية لطرفي المعادلة (26) بالنسبة الى y ؛

$$\frac{\partial g}{\partial y} = 2s\, e^{-s^2+2sy} = \sum_{n=0}^{\infty} \frac{H_n'(y)}{n!} s^n$$

$$2s \sum_{n=0}^{\infty} \frac{H_n(y)}{n!} s^n = \sum_{n=0}^{\infty} \frac{H_n'(y)}{n!} s^n$$

$$2 \sum_{n=0}^{\infty} \frac{H_n(y)}{n!} s^{n+1} = \sum_{n=0}^{\infty} \frac{H_n'(y)}{n!} s^n$$

بمساواة معاملات S^n

$$2 \sum_{n=0}^{\infty} \frac{H_{(n-1)}(y)}{\cancel{(n-1)!}} \cancel{s^n} = \sum_{n=0}^{\infty} \frac{H_n'(y)}{n\cancel{(n-1)!}} \cancel{s^n}$$

الطرفين * الوسطين

$$2n\, H_{n-1}(y) = H_n'(y) \qquad (27)$$

وإذا أخذنا الآن المشتقة الجزئية لطرفي المعادلة (26) بالنسبة لـ s نحصل على:

$$\frac{\partial g}{\partial s} = (-2s+2y)\, e^{-s^2+2ys} = \sum_{n=0}^{\infty} \frac{(-2s+2s)\, H_n(y)\, s^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{H_n(y)}{(n-1)!} s^{n-1}$$

وبمساواة معاملات s^n في طرفي المعادلة اعلاه نحصل على!

$$H_{n+1}(y) = 2y\, H_n(y) - 2n\, H_{n-1}(y) \qquad (28)$$

وباستخدام المعادلتين (27) و(28) نحصل على ما يلي!

$$2(n+1) H_n(y) = H'_{n+1}(y) = 2n H'_n(y) + 2H_n(y) - 2n H'_{n-1}(y)$$

$$= 2y H'_n(y) + 2H_n(y) - H''_n(y)$$

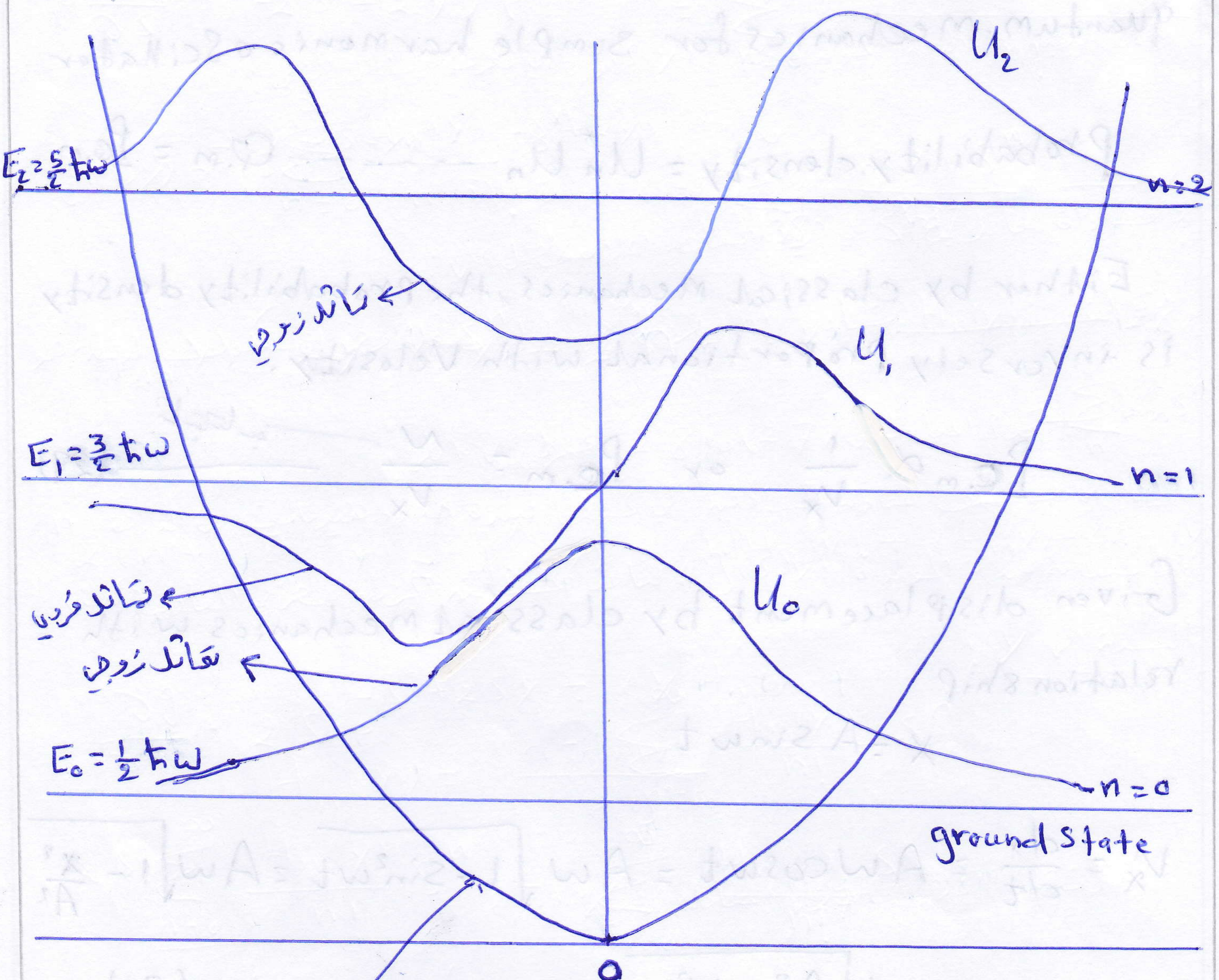
$$H''_n(y) = \frac{d^2 H_n(y)}{dy^2}$$

وفي النتيجة اعلاه نحصل على المعادلة الآتية!

$$\boxed{H''_n - 2y H'_n + 2n H_n = 0}$$

ارسم بعض الدوال الموجية

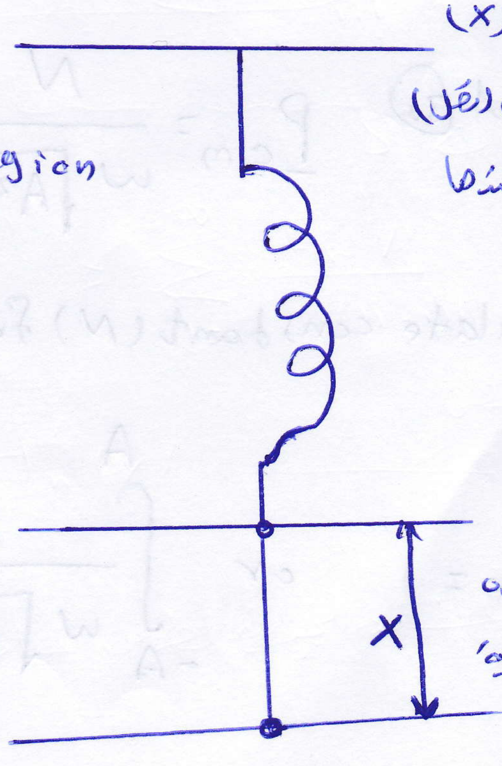
(3-6) Draw some wave Function



منطقة محرمة (ممنوعة) كلاسيكياً

Classian forbbeden region

* الطاقة الكامنة تعتمد على (x)
الدالة الموجية المرسومة تمثل
الأحوال الأدنى وتكون ...



لا يمكن ان يتجاوز هذه
المنطقة في الناحية الكلاسيكية

Comparision between the results of classical and quantum mechanics for simple harmonic oscillator

$$\text{Probability density} = U_n^* U_n \quad \text{------} \quad Q.m = P_{Q.m}$$

Either by classical mechanics, the probability density is inversely proportional with Velosity.

$$P_{c.m} \propto \frac{1}{V_x} \quad \text{or} \quad P_{c.m} = \frac{N}{V_x} \text{ ——— } (29)$$

كثابت

Given displacement by classical mechanics with relationship

$$X = A \sin \omega t$$

$$V_x = \frac{dx}{dt} = A\omega \cos \omega t = A\omega\sqrt{1-\sin^2 \omega t} = A\omega\sqrt{1-\frac{x^2}{A^2}}$$

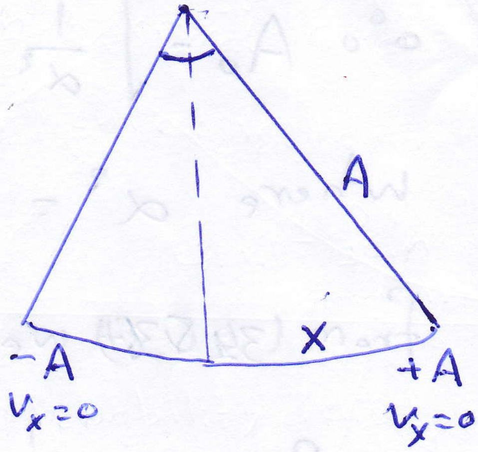
$$= \omega\sqrt{A^2 - x^2} \text{ ——— } (30)$$

From (29) and (30)
$$P_{c.m} = \frac{N}{\omega\sqrt{A^2-x^2}} \text{ ——— } (31)$$

We can calculate constant (N) from normalization condition:

$$\int_{-A}^{A} P_{c.m}\, dx = 1 \quad \text{or} \quad \int_{-A}^{A} \frac{N}{\omega\sqrt{A^2-x^2}}\, dx = 1$$

or $\frac{N}{\omega}\left[\sin^{-1}\frac{x}{A}\right]_{-A}^{A} = 1$



$$\Rightarrow \frac{N}{\omega}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

$$\frac{N\pi}{\omega} = 1 \Longrightarrow \boxed{N = \frac{\omega}{\pi}}$$

$$\therefore P_{c.m}(\text{normalized}) = \frac{1}{\pi\sqrt{A^2 - x^2}} \quad \text{——— (32)}$$

In classical mechanics, total energy given by:

$$E = \frac{1}{2} m \omega^2 A^2$$

$$\Rightarrow A = \sqrt{\frac{2E}{m\omega^2}} \quad \text{——— (33)}$$

سوف نجعل طاقة المتذبذب متساوية ومسموحة حسب الميكانيك الكمي وتكون مسموحة
ايضاً حسب الميكانيك الكلاسيكي .

$$E = E_0 \pm \frac{1}{2}\hbar\omega$$

لنأخذ الطاقة الكلية

$$\therefore A = A_0 = \sqrt{\frac{2E}{m\omega^2}}$$

$$= \sqrt{\frac{2 \times \frac{1}{2}\hbar\omega}{m\omega^2}}$$

$$= \sqrt{\frac{\hbar}{m\omega}} \quad \text{——— (34)}$$

$\therefore A_0 = \sqrt{\frac{1}{\alpha^2}} = \frac{1}{\alpha}$

where $\alpha^2 = \frac{m w}{\hbar}$

from (32 & 34) we obtain:

$$P_{C.m} = \frac{1}{\pi \sqrt{A_0^2 - x^2}} = \frac{1}{\pi \sqrt{\frac{1}{\alpha^2} - x^2}} \quad \text{—— (35)}$$

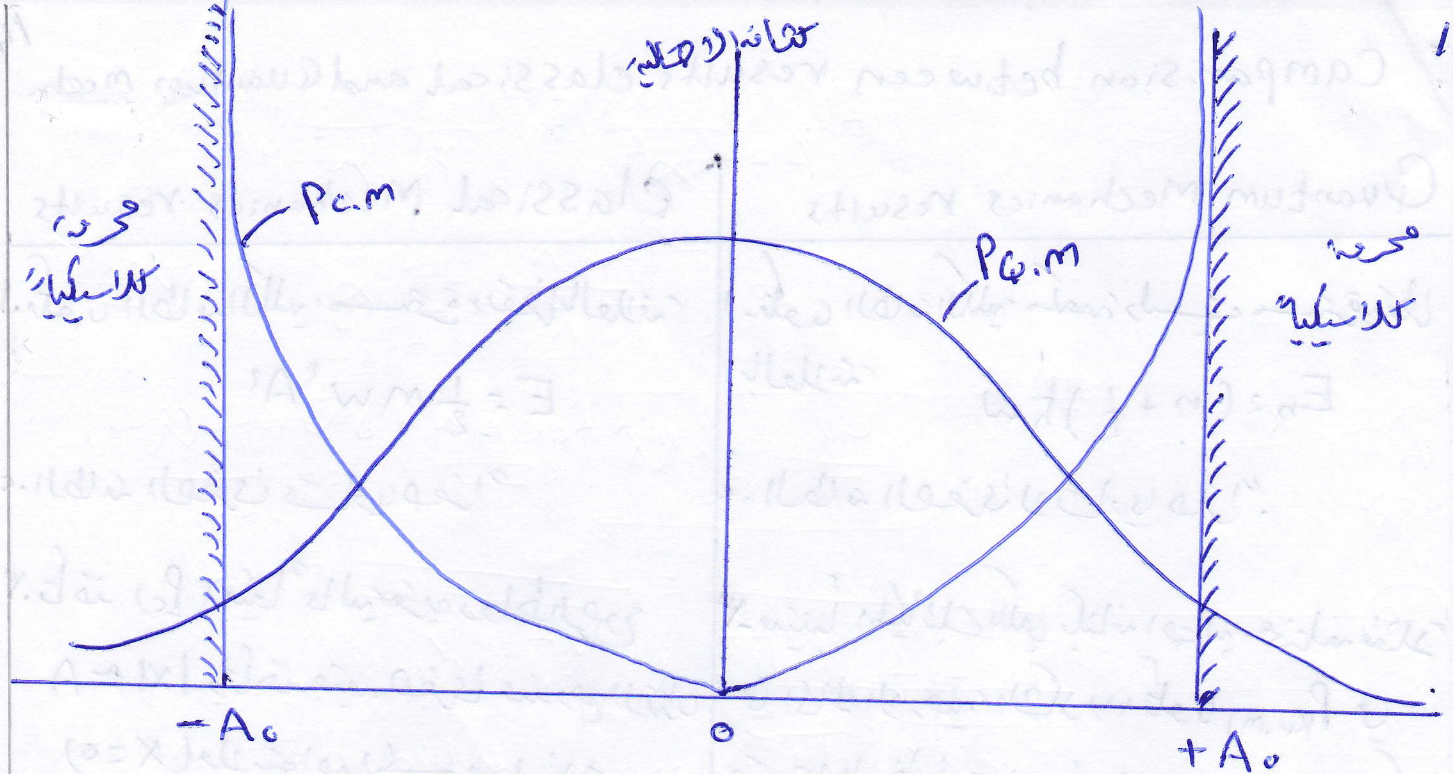
Given probability density by relation:

$$P_{Q.m} = U_0^* U_0 = \left(\frac{\alpha}{\sqrt{\pi}}\right) e^{-\alpha^2 x^2} \quad \text{—— Q.m}$$

$$P_{Q.m}\ |x| > A_0 = 2 \int_{A_0}^{\infty} U_0^2\, dx \cong 0.16 = 16\% \cong 15.8\%$$

احتمال تواجد الجسم في المنطقة المحرمة كلاسيكيا

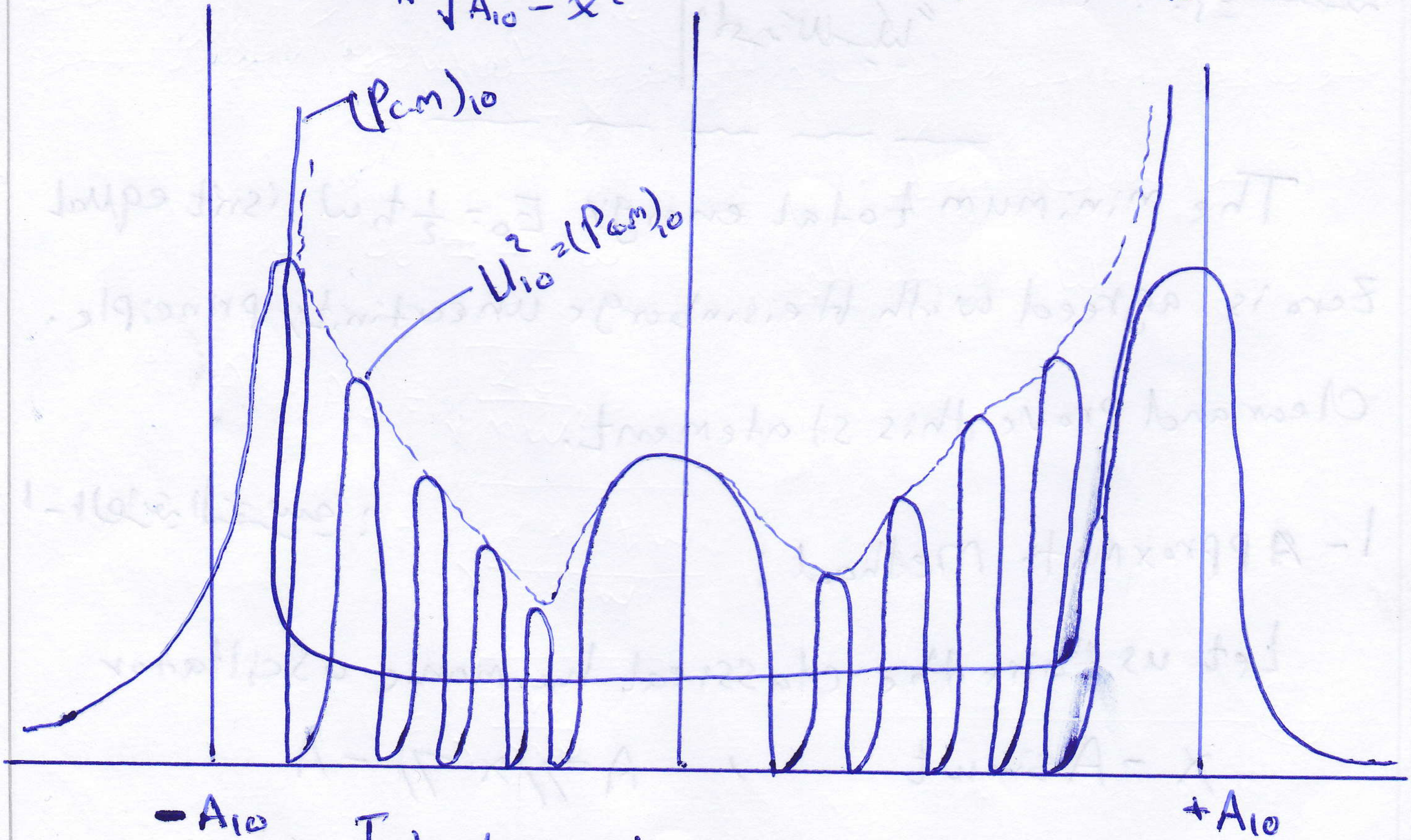
جاءت لانها دالة زوجية



Where $E = E_{10} = (10 + \frac{1}{2})\hbar\omega = \frac{21}{2}\hbar\omega$

$$A_{10} = \sqrt{\frac{2E_{10}}{m\omega^2}} \simeq \frac{\sqrt{21}}{\alpha} \simeq \frac{4.5}{\alpha} = 4.5\,A_0$$

$$(P_{c.m})_{10} = \frac{1}{\pi\sqrt{A_{10}^2 - x^2}} \Longrightarrow U_{10}^2 = (P_{c.m})_{10}$$



Introduction to Q.M Pauling and Wilson

Camparision between results classical and Quantum mech.:

Quantum Mechanics results	Classical Mechanics results
١- تكون الطاقة الكلية مكممة وليست مستمرة وتعطى بالعلاقة $E_n = (n+\frac{1}{2})\hbar\omega$	١. تكون الطاقة الكلية مستمرة وتعطى بالعلاقة $E = \frac{1}{2} m\omega^2 A^2$
٢- الطاقة الصغرى لا تساوي صفراً".	٢. الطاقة الصغرى تساوي صفراً"
٣- مبدأ الميكانيك الكمي بكثافة الاحتمالية مختلفة فمثلاً في الحالة الارضية (الشكل ١) تكون $P_{(Q.M)}$ في قيمتها العظمى عند موضع الاتزان $(x=0)$ وتكون قيمتها صغيرة ونوجد احتمالية لتواجد الجسم في المنطقة المحرمة كلاسيكياً $(\lvert x\rvert > A)$. ولكن بزيادة الطاقة فان $P_{(Q.M)}$ تقترب وتصبح بمعدل لـ P_c $P_{Q.M}$ عند $(n=10)$ مثلاً" (الشكل) وتقل احتمالية تواجد الجسم في المنطقة المحرمة كلاسيكياً"	٣. تأخذ (P_c) قيماً عالية قرب نقاط الرجوع $\lvert x\rvert \simeq A$ وتأخذ قيمة صغرى عند موضع الاتزان $(x=0)$ ولا يتواجد الجسم في المنطقة $(A<\lvert x\rvert)$

The minimum total energy $E_0 = \frac{1}{2}\hbar\omega$ isn't equal Zero is agreed with Heisinberge uncertinity principle. Clear and prove this statement.

١- الطريقة التقريبية:

1- Approxmate Method:

Let us take the classical harmonic oscillator

$x = A\sin\omega t \quad \because, \quad A \geq x \geq -A$

$\therefore \Delta x = 2A$

$$\frac{1}{2} m\omega^2 A^2 \geq \frac{\hbar\omega}{8} \quad \text{or} \quad E \geq \frac{\hbar\omega}{8}$$

$$\frac{h\omega}{8} = \frac{2\pi}{8}\hbar\omega = \frac{2\pi}{4}\overbrace{\left(\frac{1}{2}\hbar\omega\right)}^{E_0} = 1.5\,E_0$$

$$E \geq 1.5\,E_0$$

Its similar to the result which obtain from Shrodinger equation soultion, or, $E \geq E_0$ which mean that the energy didn't begin from zero.

2- Exact method; الطريقة المضبوطة

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

لنأخذ مثلاً الحالة الأرضية التي دالتها الموجية

$$U_0 = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{\frac{1}{2}} e^{-\alpha^2 x^2/2}$$

$$\langle x \rangle = \int U_0^* \hat{x}\, U_0\, dx = \int_{-\infty}^{\infty} x\, U_0^2\, dx$$

$$= N_0^2 \int_{-\infty}^{\infty} x\, e^{-\alpha^2 x^2}\, dx = 0$$ لأن الدالة فردية

$$\langle x^2 \rangle = \int U_0^* \hat{x}^2 U_0\, dx = \int_{-\infty}^{\infty} x^2 U_0^2\, dx$$

$$= N_0^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2}\, dx$$