CHAPTER TWO (GAUSS'S LAW)

In the preceding chapter we showed how to use Coulomb's law to calculate the electric field generated by a given charge distribution. In this chapter, we describe Gauss's law and an alternative procedure for calculating electric fields.

ELECTRIC FLUX

Consider an electric field that is uniform in both magnitude and direction, as shown in Figure 2.1. The product of the magnitude of the electric field E and surface area A perpendicular to the field is called the **electric flux** Φ_E

$$\Phi_E = EA \qquad (2.1)$$

From the SI units of E and A, we see that Φ_E has units of ($N.m^2/C$). **Electric flux is proportional to the number of electric field lines penetrating some surface.**

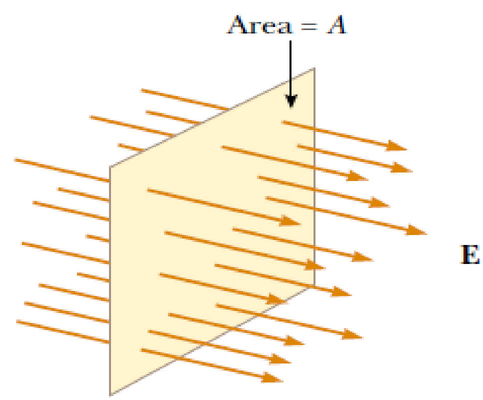

Figure 2.1 Field lines representing a uniform electric field penetrating a plane of area A perpendicular to the field. The electric flux Φ_E through this area is equal to EA.

EXAMPLE1:

What is the electric flux through a sphere that has a radius of 1.00 m and carries a charge of +1.00 µC at its center?

Solution The magnitude of the electric field 1.00 m from this charge is given by equation

$E = \frac{k\,|q|}{r^2} = (9 \times 10^9\ N.m^2/C^2)\ (1.0 \times 10^{-6}\ C)/\ (1.00\ m)^2$

$E_1 = 9 \times 10^3$ N/C

The field points radially outward and is therefore everywhere perpendicular to the surface of the sphere. The flux through the sphere (whose surface area $A = 4\pi r^2 = (12.56\ m^2)$ is thus

$$\Phi_E = EA = (9\times10^3\ \mathrm{N/C})\ (12.56\ \mathrm{m^2})$$
$$= 1.13\times10^5\ \mathrm{N.m^2/C}$$

Home work: What would be the (a) electric field and (b) flux through the sphere if it had a radius of 0.500 m?

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If the surface under consideration is not perpendicular to the field, the flux through it must be less than that given by Equation 2.1. From Figure 2.2 we see that the two areas are related by $A' = A\cos\theta$. Because the flux through A equals the flux through A', we conclude that the flux through A is

$\Phi_E = E A' = EA\cos\theta$

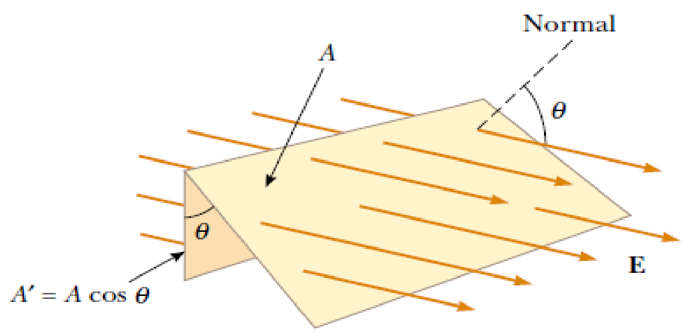


Figure 2.2 Field lines representing a uniform electric field penetrating an area A that is at an angle θ to the field. Because the number of lines that go through the area A' is the same as the number that go through A, the flux through A' is equal to the flux through A and is given by :
$\Phi_E = EA\cos\theta$

From this result, we see that the flux through a surface of fixed area A has a maximum value EA when the surface is perpendicular to the field (in other words,when the normal to the surface is parallel to the field, that is, θ=0 in Figure 2.2); the flux is zero when the surface is parallel to the field (in other words, when the normal to the surface is perpendicular to the field, that is, θ=90. In more general consider a surface divided up into a large number of small elements (Figure 2.3)

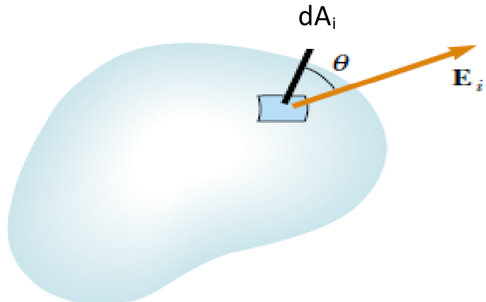
$$\Phi_E = \int_{surface} E.dA \qquad \text{(Definition of electric flux)}$$

The net flux through the surface is proportional to the net number of lines leaving the surface. If more lines are leaving than entering, the net flux is positive. If more lines are entering than leaving, the net flux is negative.
we can write the net flux Φ_E through a closed surface as

$$\boldsymbol{\Phi_E = \int E.dA = \int E_n dA} \qquad \textbf{(2.2)}$$

Where E_n represents the component of the electric field normal to the surface.

Figure 2.3 a small element of surface area dA_i . The electric field makes an angle θ with the vector dA_i , defined as being normal to the surface element, and the flux through the element is equal to $E_i\ dA_i \cos$.



EXAMPLE 2:-Consider a uniform electric field E oriented in the x direction. Find the net electric flux through the surface of a cube of edges L, oriented as shown in Figure 2.4.

Solution : The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (3),(4), and the unnumbered ones) is zero because E is perpendicular to dA on these faces.

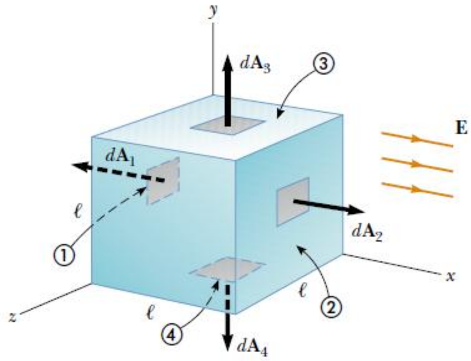


Figure 2.4 A closed surface in the shape of a cube in a uniform electric field oriented parallel to the x axis.

The net flux through faces 1and 2 is:

$$\boldsymbol{\Phi_E = \int_1 E.dA + \int_2 E.dA}$$

For face(1), E is constant and directed inward but dA_1 is directed Outward(θ=180); thus, the flux through this face is:

$$\boldsymbol{\int_1 E.dA = \int_1 E\,(\cos 180)\,dA = -E\int_1 dA = -EA = -EL^2}$$

Because $\mathbf{L^2}$ the area of each face is $A=L^2$

For face(2) ,E is constant and outward and in the same direction as dA_2(θ=0); hence, the flux through this face is:

$$\boldsymbol{\int_2 E.dA = \int_2 E\,(\cos 0)\,dA = E\int_2 dA = +EA = EL^2}$$

Therefore, the net flux over all six faces is:

$$\boldsymbol{\Phi_E = -EL^2 + EL^2 + 0 + 0 + 0 + 0 = 0}$$

GAUSS'S LAW

Consider a positive point charge q located at the center of a sphere of radius r, as shown in Figure 2.5. From Equation ($E= k\,|q|\,/r^{2}$) we know that the magnitude of the electric field everywhere on the surface of the sphere .

As noted in Example 2.1, the field lines are directed radially outward and hence perpendicular to the surface at every point on the surface. That is, at each surface point, E is parallel to the vector dA_i representing a local element of area ΔA_i surrounding the surface point. Therefore,

$$E_i \,.\, dA_i = E_i\, dA_i$$

and from Equation 2.2 we find that the net flux through the gaussian surface is

$$\Phi_E = \int E.dA = \int E dA = E\int dA$$

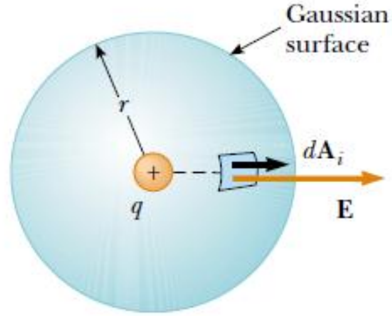


Figure 2.5 A spherical Gaussian surface of radius *r* surrounding a point charge *q*. When the charge is at the center of the sphere, the electric field is everywhere normal to the surface and constant in magnitude

where we have moved E outside of the integral because, by symmetry, E is constant over the surface and given by E= k q/r ^{2}.Furthermore, because the surface is spherical, Hence, the net flux through the gaussian surface is

$$\Phi_E = \frac{k\, q\, (4\pi\, r^2)}{r^2} = 4\,\pi\, k\, q = \frac{q}{\varepsilon_o} \quad \rightarrow \quad (\text{because } k = 1/4\pi\varepsilon_o)$$

We can verify that this expression for the net flux gives the same result as Example 1:

Notes:-

1- the net flux through any closed surface is:

$$\Phi_E = \oint_S \mathrm{E.\,dA} = \frac{q}{\varepsilon_o} \qquad \text{(Gauss's law)}$$

2- the net electric flux through a closed surface that surrounds no charge is zero.

3- the electric field due to many charges is the vector sum of the electric fields produced by the individual charges

$$\int E.dA = \int (E_1+E_2+\ldots\ldots\ldots).dA$$

5- Gauss's law is useful for evaluating E when the charge distribution has high symmetry

6- The net electric flux through any closed surface depends only on the charge inside that surface (see figure 2.6).

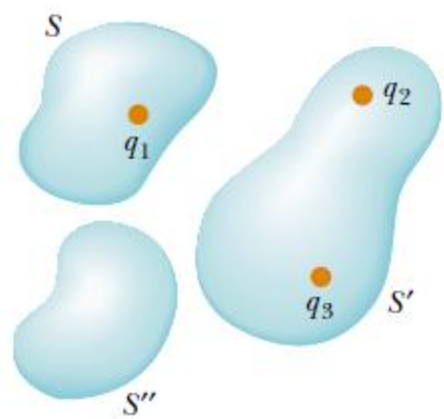


Figure 2.6 The net flux through surface S is **q_1/ε_o** ,the net flux through surface S′ is **q_2+q_3/ε_o** and the net flux through surface S″ is zero.

EXAMPLE 2.3 : A spherical gaussian surface surrounds a point charge q. Describe what happens to the total flux through the surface if (**a**) the charge is tripled, (**b**) the radius of the sphere is doubled, (**c**) the surface is changed to a cube, and (**d**) the charge is moved to another location inside the surface.

Solution (a) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(b) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(c) The flux does not change when the shape of the Gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

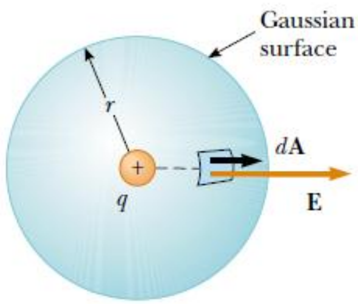
(d) The flux does not change when the charge is moved to another location inside that surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

EXAMPLE 2.4:- Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

Solution : We choose a spherical gaussian surface of radius r centered on the point charge, as shown in Figure 2.7 .The electric field due to a positive point charge is directed radially outward by symmetry and is therefore normal to the surface at every point(E is parallel to dA at each point). Therefore, E.dA = EdA and Gauss's law gives

$$\Phi_E = \oint_S \mathbf{E}.d\mathbf{A} = \oint_S E\,dA = \frac{q}{\varepsilon_o}$$

Figure 2.7 The point charge q is at the center of the spherical gaussian surface, and E is parallel to dA at every point on the surface.



By symmetry, E is constant everywhere on the surface, so it can be removed from the integral. Therefore,

$$\oint_S E\,dA = E \oint_S dA = E\,(4\pi\, r^{2}) = \frac{q}{\varepsilon_o}$$

$$E = \frac{q}{(4\pi\, r^2)\, \varepsilon_o}$$

$$E = k\, q\, / r^2$$

This is the familiar electric field due to a point charge that we developed from Coulomb's law in Chapter 1.

EXAMPLE 2.5:- An insulating solid sphere of radius **a** has a uniform volume charge density ρ and carries a total positive charge Q (Fig. 2.8).

(a) Calculate the magnitude of the electric field at a point outside the sphere.

Solution : Because the charge distribution is spherically symmetric, we again select a spherical gaussian surface of radius r, concentric with the sphere, as shown in Figure 2.7a. for the point charge in Example 2.4. we find that

$$E = k\, Q / r^2 \quad \rightarrow \quad (\text{for } r > a)$$

Note that this result is identical to the one we obtained for a point charge. Therefore, we conclude that, for a uniformly charged sphere, the field in the region external to the sphere is *equivalent* to that of a point charge located at the center of the sphere.

(b) Find the magnitude of the electric field at a point inside the sphere.

Solution: In this case we select a spherical gaussian surface having radius ($r < a$), concentric with the insulated sphere (Fig. 2.8b). Let us denote the volume of this smaller sphere by V'. To apply Gauss's law in this situation, it is important to recognize that the charge q_{in} within the Gaussian surface of volume V' is less than Q. To calculate q_{in}, we use the fact that $q_{in} = \rho\, V'$

$$q_{in} = \rho\, V' = \rho\, (\tfrac{4}{3}\pi\, r^3)$$

By symmetry, the magnitude of the electric field is constant everywhere on the spherical gaussian surface and is normal to the surface at each point. Therefore, Gauss's law in the region $r < a$ gives

$$\oint_s E\, dA = E \oint_s dA = E(4\pi\, r^2) = \frac{q_{in}}{\varepsilon_o}$$

Solving for *E* gives

$$E = \frac{q_{in}}{4\ \pi r^2 \varepsilon_o} = \frac{\rho\ (\frac{4}{3}\pi\, r^3)}{4\ \pi r^2 \varepsilon_o} = \frac{\rho\ \ r}{3\varepsilon_o}$$

Because $\rho = Q / (\frac{4}{3}\pi\ a^3)$ by definition and since $k = 1/4\pi\varepsilon_o$, this expression for *E* can be written as:

$$E = \frac{Qr}{4\ \pi a^3 \varepsilon_o} = \frac{k\ Q\ r}{a^3} \rightarrow \text{(for r <a)}$$

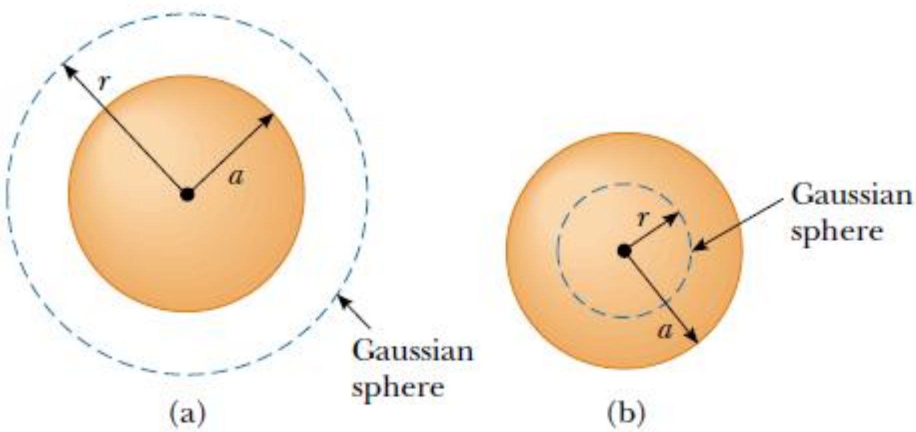


Figure 2.8 A uniformly charged insulating sphere of radius *a* and total charge *Q*. (a) The magnitude of the electric field at a point exterior to the sphere is kQ/r^2 (b) The magnitude of the electric field inside the insulating sphere is due only to the charge within the gaussian sphere is $k Q\ r / a^3$.

EXAMPLE 2.6:- A thin spherical shell of radius **a** has a total charge ***Q*** distributed uniformly over its surface (Fig. 2.9). Find the electric field at points (a) outside and (b) inside the shell.

Solution: (a) The calculation for the field outside the shell is identical to that for the solid sphere shown in Example 2.5a. Therefore,

$$E = k\,Q / r^2 \quad \rightarrow \text{(for r >a)}$$

(b) The electric field inside the spherical shell is zero. Because of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero .

Shows that E = 0 in the region r < a. We obtain the same results using Equation ($E = k\int dq/r^2$) from chapter one and integrating over the charge distribution (Gauss's law allows us to determine these results in a much simpler way).

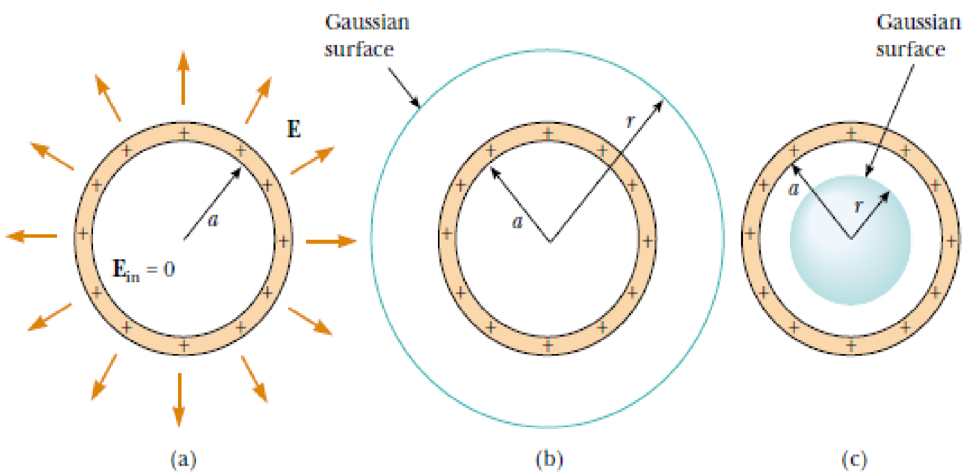


Figure 2.9 (a) The electric field inside a uniformly charged spherical shell is zero. The field outside is the same as that due to a point charge *Q* located at the center of the shell. (b) Gaussian surface for $r > a$. (c) Gaussian surface for $r < a$.

EXAMPLE 2.7:- Find the electric field a distance *r* from a line of positive charge of infinite length and constant charge per unit length λ (Fig. 2.10).

Solution : we select a cylindrical gaussian surface of radius *r* and length L that is coaxial with the line charge. For the curved part of this surface, E is constant in magnitude and perpendicular to the surface at each point. Furthermore, the flux through the ends of the gaussian cylinder is zero because E is parallel to these surfaces.

The total charge inside our gaussian surface is λL. Applying Gauss's law we find that for the curved surface

$$\Phi_E = \oint_s \mathrm{E.\,dA} = E\oint_s \mathrm{dA} = \mathrm{EA} = \frac{q_{in}}{\varepsilon_o} = \frac{\lambda L}{\varepsilon_o}$$

The area of the curved surface A = 2 πrL ; is therefore,

$$E=\frac{\lambda}{2\,\pi\varepsilon_0 r}=2k\,\frac{\lambda}{r}$$

Thus, we see that the electric field due to a cylindrically symmetric charge distribution varies as $1/r$, whereas the field external to a spherically symmetric charge distribution varies as $1/r^2$.

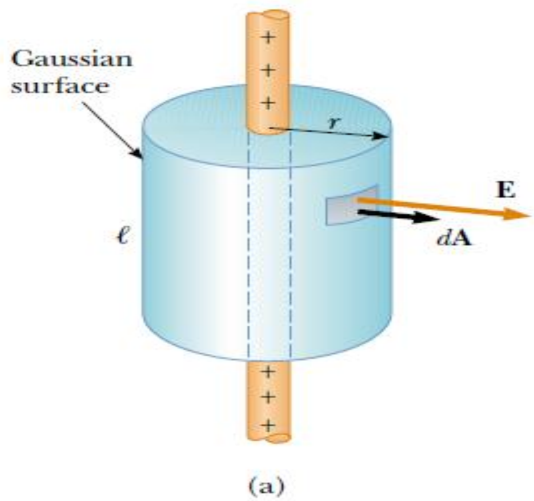


Figure 2.10 An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line.

EXAMPLE 2.8:- Find the electric field due to a nonconducting, infinite plane of positive charge with uniform surface charge density σ.

Solution : By symmetry, E must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane. The fact that the direction of E is away from positive charges indicates that the direction of E on one side of the plane must be opposite its direction on the
other side, as shown in Figure 2.11. The flux through each end of the cylinder is *EA*; hence, the total flux through the entire gaussian surface is just that through the ends, $\Phi_E = 2EA$.
Noting that the total charge inside the surface is $q_{in} = \sigma A$, we use Gauss's law and find that

$\Phi_E = 2EA = q_{in}/\varepsilon_o = \sigma A/\varepsilon_o$

$E = \sigma / 2\varepsilon_o$ →(*)

Because the distance from each flat end of the cylinder to the plane does not appear in Equation (*), we conclude that $E = \sigma / 2\varepsilon_o$ at any distance from the plane. That is, the field is uniform everywhere

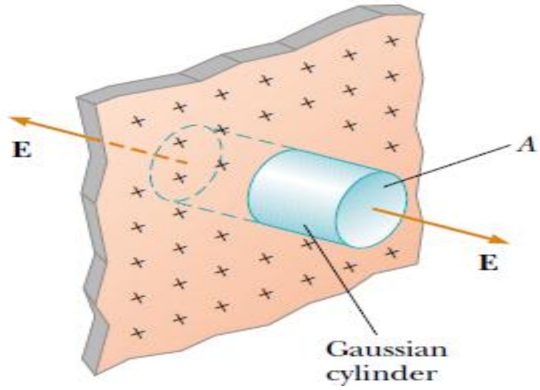


Figure 2.11 A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is *EA* through each end of the gaussian surface and zero through its curved surface.

EXAMPLE 2.9:- Explain why Gauss's law cannot be used to calculate the electric field near an electric dipole, a charged disk, or a triangle with a point charge at each corner.

Solution The charge distributions of all these configurations do not have sufficient symmetry to make the use of Gauss's law practical. We cannot find a closed surface surrounding any of these distributions.

CONDUCTORS IN ELECTROSTATIC EQUILIBRIUM

When there is no net motion of charge within a conductor, the conductor is in electrostatic equilibrium. As we shall see, a conductor in electrostatic equilibrium has the following properties:

1. The electric field is zero everywhere inside the conductor.
2. If an isolated conductor carries a charge, the charge resides on its surface.
3. The electric field just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ / ε_o, where σ is the surface charge density at that point ($\Phi_E = \int E.dA = EA = q_{in}/\varepsilon_o = \sigma A/\varepsilon_o$ where we have used the fact that $q_{in} = \sigma A$. Solving for E gives $E = \sigma / \varepsilon_o$).

4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

EXAMPLE 2.10:- A solid conducting sphere of radius ***a*** carries a net positive charge 2*Q*. A conducting spherical shell of inner radius ***b*** and outer radius ***c*** is concentric with the solid sphere and carries a net charge -*Q*. Using Gauss's law, find the electric field in the regions labeled (1),(2),(3) and(4) in Figure 2.12 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

Solution : To determine the electric field at various distances *r* from this center, we construct a spherical gaussian surface for each of the four regions of interest.

To find *E* inside the solid sphere (region (1)), consider a gaussian surface of radius ***r <a***. Because there can be no charge inside a conductor in electrostatic equilibrium, we see that $q_{in} = 0$; thus, on the basis of Gauss's law and symmetry $E_1=0$, for ***r<a***.

In region (2)between the surface of the solid sphere and the inner surface of the shell—we construct a spherical gaussian surface of radius *r* where *a <r < b* and note that the charge inside this surface is +2*Q*. Because of the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on the gaussian surface. Following Example 2.4 and using Gauss's law, we find that:

$$E_2A = E_2(4\pi r^2) = q_{in}/\varepsilon_o = 2Q/\varepsilon_o \text{ (for } a < r < b)$$

(In region(4), where *r* > *c*, the spherical gaussian surface we construct surrounds a total charge of $q_{in}=2Q+(-Q)=Q$. Therefore, application of Gauss's law to this surface gives

$$E_4 = kQ/r^2 \quad \text{(for } r > c)$$

In region (3), the electric field must be zero because the spherical shell is also a conductor in equilibrium. If we construct a gaussian surface of radius *r* where *b< r < c*, we see that q_{in} must be zero because $E_3= 0$ From this argument, we conclude that the charge on the inner surface of the spherical shell must be -2*Q* to cancel the charge +2*Q* on the solid sphere. Because the net charge on the shell is -*Q*, we conclude that its outer surface must carry a charge +*Q*.

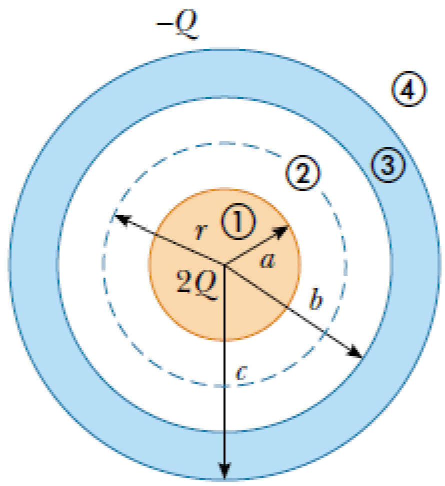


Figure 2.12 A solid conducting sphere of radius *a* and carrying a charge 2*Q* surrounded by a conducting spherical shell carrying a charge -*Q*.