Chapter 1

Geometric Optics

1.1 Number of wavelengths between two points. A light wave of vacuum wavelength 500 nm travels from point A to point B that is a distance of 0.01 m away.

(1) If the space containing points A and B is vacuum, how many wavelengths span the space from A to B?

(2) If the region between points A and B is completely filled with glass of refractive index $n = 1.5$, how many wavelengths span the space from A to B?

Let D_{AB} be the distance between points A and B, λ the vacuum wavelength, and λ' the wavelength in glass.

(1) In vacuum, the number of wavelengths that span the space from A to B is

$$N = \frac{D_{AB}}{\lambda} = \frac{0.01}{500 \times 10^{-9}} = 2 \times 10^4.$$

(2) Note that the frequency of a light beam remains unchanged no matter in which medium it travels. The wavelength of light in glass is given by

$$\lambda' = \frac{v}{f'} = \frac{c/n}{f} = \frac{c/f}{n} = \frac{\lambda}{n},$$

where v is the speed of light in glass and f and f' are respectively the frequencies of light in vacuum and in glass with $f' = f$. If the region between points A and B is completely filled with glass, the number of wavelengths that span the space from A to B is given by

$$N' = \frac{D_{AB}}{\lambda'} = n\frac{D_{AB}}{\lambda} = nN = 1.5 \times 2 \times 10^4 = 3 \times 10^4.$$

1.2 Dispersion of fused silica. The wavelength-dependence of the refractive index of fused silica is given by

$$n^2 = 1 + \frac{B_1\lambda^2}{\lambda^2 - C_1} + \frac{B_2\lambda^2}{\lambda^2 - C_2} + \frac{B_3\lambda^2}{\lambda^2 - C_3},$$

where $B_1 = 0.696$, $B_2 = 0.407$, $B_3 = 0.897$, $C_1 = 4.679 \times 10^3\ \text{nm}^2$, $C_2 = 1.351 \times 10^4\ \text{nm}^2$, and $C_3 = 9.793 \times 10^7\ \text{nm}^2$.

(1) At what wavelengths, does n diverge? If n is infinite for light of a particular wavelength, can the light of this wavelength propagate in fused silica?

(2) At what wavelengths, is n equal to unity?

(3) What is the value of n as $\lambda \to \infty$?

(4) Plot n as a function of λ in the visible region for λ from 400 nm to 700 nm.

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(1) From the given expression of n in terms of λ, we see that n diverges at $\lambda \approx 6.840 \times 10^1$, 1.162×10^2, and 9.896×10^3 nm. If $n = \infty$ for light of a particular wavelength, then $v = c/n = 0$, which implies that the light of this wavelength can not propagate in fused silica and will be absorbed.

(2) To find the wavelengths at which $n = 1$, we set the given expression for n^2 to unity and obtain

$$\lambda^2 \left(\frac{B_1}{\lambda^2 - C_1} + \frac{B_2}{\lambda^2 - C_2} + \frac{B_3}{\lambda^2 - C_3} \right) = 0.$$

From the above equation, we see that $n = 1$ at $\lambda_1 = 0$. The other wavelengths at which $n = 1$ are to be solved from

$$\frac{B_1}{\lambda^2 - C_1} + \frac{B_2}{\lambda^2 - C_2} + \frac{B_3}{\lambda^2 - C_3} = 0.$$

The above equation is actually a quadratic algebraic equation in λ^2. Simplifying the above equation, we have

$$\alpha\lambda^4 + \beta\lambda^2 + \gamma = 0, \tag{1.1}$$

where

$$\begin{aligned}
\alpha &= B_1 + B_2 + B_3, \\
\beta &= -\big[B_1(C_2+C_3)+B_2(C_3+C_1)+B_3(C_1+C_2)\big], \\
\gamma &= B_1C_2C_3 + B_2C_3C_1 + B_3C_1C_2.
\end{aligned}$$

Solving for λ^2 from Eq. (1.1), we obtain

$$\lambda^2 = \frac{1}{2\alpha}\left[-\beta \pm \left(\beta^2 - 4\alpha\gamma\right)^{1/2} \right]. \tag{1.2}$$

Evaluating the above expression using the given values of constants, we obtain the following two positive solutions for λ

$$\lambda_2 = 1.013 \times 10^2 \text{ nm}, \quad \lambda_3 = 7.349 \times 10^3 \text{ nm}. \tag{1.3}$$

In summary, the wavelengths at which $n = 1$ are

$$\lambda_1 = 0, \tag{1.4a}$$

$$\lambda_2 = 1.013 \times 10^2 \text{ nm}, \tag{1.4b}$$

$$\lambda_3 = 7.349 \times 10^3 \text{ nm}. \tag{1.4c}$$

(3) As $\lambda \to \infty$, n^2 is given by

$$n^2 = 1 + B_1 + B_2 + B_3 = 3.$$

Thus, $n \to 1.732$ as $\lambda \to \infty$.

(4) The plot of n as a function of λ in the visible region for λ from 400 nm to 700 nm is given in Fig. 1.1 from which we see that n is a monotonically decreasing function of λ in the visible region.

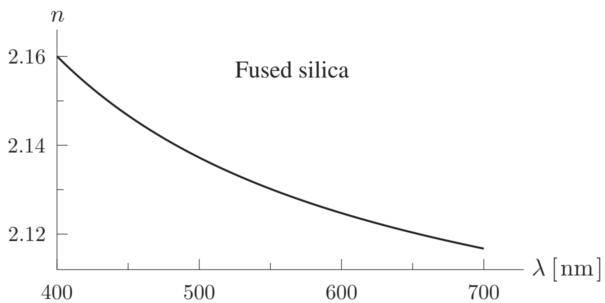

Fig. 1.1: Refractive index of fused silica in the visible region.

1.3 Spread of the components of a light ray through a prism. Consider the passage of a light ray with two monochromatic components through a trihedral prism with refracting angle α. The refractive indices of the prism with respect to the two components are respectively n and $n + \Delta n$ with Δn small. The prism is so oriented that the deflection angle is minimum. Find the angle $\Delta\theta$ between the two components of the light ray after its passing through the prism.

Let θ_i be the angle of incidence into the prism. Let θ be the angle of refraction out of the prism corresponding to the refractive index n. We have

$$\gamma = \theta_i + \theta - \alpha.$$

From the above equation, we have

$$\Delta\theta = \Delta\gamma.$$

For the minimum deflection, we also have

$$\sin\frac{\gamma+\alpha}{2} = n\sin\frac{\alpha}{2}.$$

Differentiating the above equation with respect to n, we have

$$\frac{1}{2}\cos\frac{\gamma+\alpha}{2}\frac{d\gamma}{dn} = \sin\frac{\alpha}{2},$$

$$\Delta\gamma \approx \frac{2\sin(\alpha/2)}{\cos[(\gamma+\alpha)/2]}\Delta n = \frac{2\sin(\alpha/2)}{\sqrt{1-n^2\sin^2(\alpha/2)}}\Delta n.$$

We thus have

$$\Delta\theta = \Delta\gamma = \frac{2\sin(\alpha/2)}{\sqrt{1-n^2\sin^2(\alpha/2)}}\Delta n. \tag{1.5}$$

1.4 Dispersion on an equilateral prism. A beam of white light is incident on an equilateral prism [Ref. Fig. 1.2] made of glass with $n = 1.67$ at $\lambda = 400$ nm and $n = 1.61$ at $\lambda = 750$ nm. Assume that the red rays in the prism are parallel to the bottom face. What is the angle between the transmitted red and violet beams?

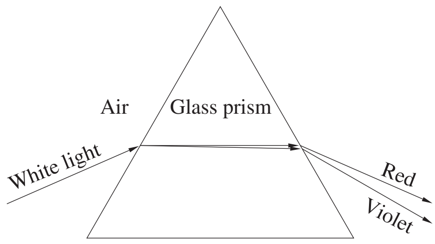

Fig. 1.2: Dispersion of white light on an equilateral prism.

Shown in Fig. 1.3 is the ray diagram for light of a certain wavelength for which the refractive index of the prism is n. The angles of incidence and refraction at the entrance point are denoted respectively by θ_1 and θ_2; they are denoted respectively by θ_1' and θ_2' at the exit point.

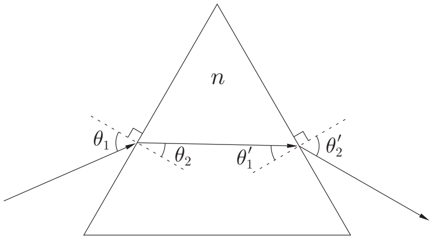

Fig. 1.3: Ray diagram for light for which the refractive index of the prism is equal to n.

Since the red light in the prism propagates parallel to the bottom face of the prism, the angle of refraction at the entrance point is given by $\theta_2 = 30^\circ$. From the law of refraction $n_1 \sin\theta_1 = n_2 \sin\theta_2$, we have

$$\theta_1 = \arcsin\left(\frac{n_2 \sin\theta_2}{n_1}\right) \approx \arcsin\left(\frac{1.61 \times \sin 30^\circ}{1.00}\right) \approx 53.61^\circ, \quad (1.6)$$

where we have taken the refractive index of air to be approximately unity.

From the geometry in Fig. 1.3, we can infer that

$$\theta_1' = 60^\circ - \theta_2. \quad (1.7)$$

For light for which the refractive index of the prism is n, the angle of refraction θ_2' at the exit point is given by

$$\theta_2' = \arcsin(n \sin\theta_1') = \arcsin[n \sin(60^\circ - \theta_2)] \quad (1.8)$$

with θ_2 given by

$$\theta_2 = \arcsin\left(\frac{\sin\theta_1}{n}\right). \quad (1.9)$$

The angle between the transmitted red and violet beams is given by

$$\Delta\theta_2 = \theta'_{2,\text{violet}} - \theta'_{2,\text{red}}. \quad (1.10)$$

Evaluating the angles θ_2, θ_1', θ_2', and $\Delta\theta_2$ using the value of θ_1 in Eq. (1.6) and Eqs. (1.7) through (1.9), we obtain the results in Table 1.1 for red and violet beams.

From the values of θ_2' in Table 1.1 and Eq. (1.10), we have

$$\Delta\theta_2 = \theta'_{2,\text{violet}} - \theta'_{2,\text{red}} \approx 59.84 - 53.61 = 6.23^\circ. \quad (1.11)$$

1.5 Path length in a glass plate. A laser beam is incident on the top surface of a parallel glass plate of thickness t and refractive index n_g.

Table 1.1: Values of relevant angles for red and violet beams through the prism.

Beam	n	θ_2 [°]	θ_1' [°]	θ_2' [°]
Red	1.61	30.00	30.00	53.61
Violet	1.67	28.82	31.18	59.84

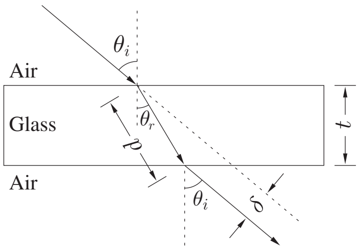

Fig. 1.4: Laser beam passing through a parallel glass plate.

Find the lateral displacement δ of the laser beam and the true length d of the path through the glass in terms of the angle of incidence θ_i, the thickness t and the refractive index n_g of the plate, and the refractive index n_a of air. What is the optical path length in the glass?

Applying the law of refraction to the refraction at the top surface, we have $n_a \sin\theta_i = n_g \sin\theta_r$. Thus,

$$\sin\theta_r = \frac{n_a \sin\theta_i}{n_g}, \quad \cos\theta_r = \frac{1}{n_g}\left(n_g^2 - n_a^2 \sin^2\theta_i\right)^{1/2}.$$

The true length d of the path is given by

$$d = \frac{t}{\cos\theta_r} = \frac{n_g t}{\left(n_g^2 - n_a^2 \sin^2\theta_i\right)^{1/2}}. \tag{1.12}$$

The lateral displacement δ of the laser beam is given by

$$\begin{aligned}\delta &= t\,(\tan\theta_i - \tan\theta_r)\cos\theta_i \\ &= t\left[1 - \frac{n_a \cos\theta_i}{\left(n_g^2 - n_a^2 \sin^2\theta_i\right)^{1/2}}\right]\sin\theta_i.\end{aligned} \tag{1.13}$$

The optical path length S in the glass is given by

$$S = n_g d = \frac{n_g^2 t}{\left(n_g^2 - n_a^2 \sin^2\theta_i\right)^{1/2}}. \tag{1.14}$$

1.6 Reflection and refraction of light on a slab of glass. Consider the reflection and refraction of monochromatic light on the interfaces between air of refractive index $n_a \approx 1.00$ and a slab of glass of refractive index $n_g = 1.50$ as shown in Fig. 1.5. The upper and lower interfaces are parallel. The angle of incidence θ_1 is 50°.

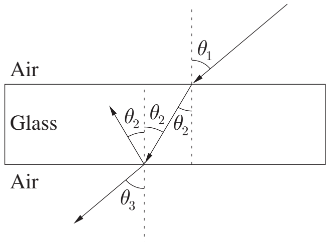

Fig. 1.5: Reflection and refraction of monochromatic light on the interfaces between air and a slab of glass.

(1) Find the angle of refraction θ_2 at the upper interface.
(2) Find the angle of refraction θ_3 at the lower interface.

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(1) From the law of refraction (Snell's law) $n_a \sin\theta_1 = n_g \sin\theta_2$, we have

$$\theta_2 = \arcsin\left(\frac{n_a \sin\theta_1}{n_g}\right) \approx \arcsin\left(\frac{1.00 \times \sin 50^{\circ}}{1.50}\right) \approx 30.71^{\circ}.$$

(2) Making use of the fact that the angle of incidence at the lower interface is equal to the angle of refraction at the upper interface, we have $n_g \sin\theta_2 = n_a \sin\theta_3$ from the law of refraction. Thus,

$$\theta_3 = \arcsin\left(\frac{n_g \sin\theta_2}{n_a}\right) \approx \arcsin\left(\frac{1.50 \times \sin 30.71^{\circ}}{1.00}\right) = 50^{\circ}.$$

That $\theta_3 = \theta_1$ is required by the fact that the light ray can be traced back. This fact can be directly used to obtain the value of θ_3.

1.7 Light propagation above the surface of the Earth. Consider the propagation of a light ray above the surface of the Earth.

(1) If the light ray is propagating in a horizontal direction above the surface of the Earth with the refractive index of air given by 1.500×10^{-7} m^{-1}, what is the radius of curvature of the light ray?

(2) For the light ray to be able to travel around the earth along a circular path, what is the value of the gradient of the refractive index along the circular path?

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(1) The refractive index n of air above the surface of the Earth depends only on the radial distance r from the center of the Earth, $n = n(r)$. Assume that the light ray travels horizontally at the radial distance R from the center of the Earth.

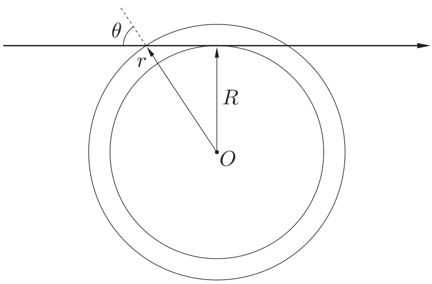

Fig. 1.6: Propagation of a light ray above the surface of the Earth.

At the radial distance r with $r \gtrsim R$, the angle θ that the light ray makes with the normal of the spherical surface of radius r is given through

$$\sin\theta = \frac{R}{r}.$$

Note that θ is the angle of incidence of the light ray onto the spherical surface of radius r. Differentiating the above equation with respect to r yields

$$\cos\theta\frac{d\theta}{dr} = -\frac{R}{r^2}. \tag{1.15}$$

The above two results will be used in the following derivations to relate θ to r.
From the law of refraction, we have

$$n\sin\theta = \text{constant}.$$

Differentiating the above equation with respect to r yields

$$\sin\theta\frac{dn}{dr} + n\cos\theta\frac{d\theta}{dr} = 0, \tag{1.16}$$

$$\frac{R}{r}\frac{dn}{dr} - n\frac{R}{r^2} = 0,$$

$$r = \frac{n}{dn/dr}. \tag{1.17}$$

For $r = R$, we have $R = n/(dn/dR)$. For $dn/dR = 1.500 \times 10^{-7}\ \mathrm{m}^{-1}$ and $n \approx 1$, we have

$$R \approx \frac{1}{1.500 \times 10^{-7}} \approx 6.667 \times 10^6\ \mathrm{m} = 6667\ \mathrm{km}. \tag{1.18}$$

(2) We return to Eq. (1.16). Inserting Eq. (1.15) into Eq. (1.16), we obtain

$$\sin\theta \frac{dn}{dr} - n\frac{R}{r^2} = 0.$$

Solving for dn/dr from the above equation yieds

$$\frac{dn}{dr} = \frac{nR}{r^2 \sin\theta}.$$

For the light ray to be able to travel around the Earth along a circular path, we must have $\sin\theta = \pi/2$. We thus have

$$\frac{dn}{dr} = \frac{nR}{r^2}.$$

For $r = R = R_{\mathrm{Earth}}$, we have

$$\frac{dn}{dR_{\mathrm{Earth}}} = \frac{n}{R_{\mathrm{Earth}}}.$$

Making use of $R_{\mathrm{Earth}} \approx 6.4 \times 10^6$ and $n \approx 1$, we have

$$\frac{dn}{dR_{\mathrm{Earth}}} \approx \frac{1}{6.4 \times 10^6} \approx 1.6 \times 10^{-7}\ \mathrm{m}^{-1}. \tag{1.19}$$

1.8 Object in water. Suppose that an object is put into a swimming pool filled with water. We consider the apparent depth of the object.

(1) Derive an expression for the apparent depth of a point on the object at an arbitrary angle of refraction from the water into air.

(2) Find the apparent depth of the object if it is looked at straight down from a position above the water.

(3) Plot the ratio of the apparent depth to the true depth as a function of the angle of refraction from the water into air.

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(1) We only consider non-internal reflections of light from the object. The reflections of light from two symmetric points of a flat object are schematically shown in Fig. 1.7.

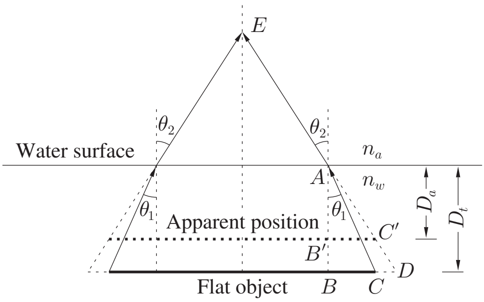

Fig. 1.7: Flat object in water.

The true depth of the object is denoted by D_t while its apparent depth is denoted by D_a. The apparent position of the object is determined by the tangentially contacting points of the raised virtual object with the extensions of light rays from the water into air as indicated by the oblique dotted lines in Fig. 1.7. Note that $\overline{B'C'} = \overline{BC}$.

Let n_w and n_a be respectively the refractive indices of water and air. The light reflected from the object will be refracted at the surface of the water before it reaches the eye.

From the right triangle $\Delta B'C'A$, we have

$$D_a = \overline{AB'} = \overline{B'C'} \cot\theta_2.$$

Making use of the fact that $\overline{B'C'} = \overline{BC}$, we have

$$D_a = \overline{BC} \cot\theta_2.$$

From the right triangle ΔBCA, we have

$$\overline{BC} = \overline{AB} \tan\theta_1 = D_t \tan\theta_1.$$

Thus,

$$D_a = D_t \tan\theta_1 \cot\theta_2.$$

From the law of refraction, we have $n_w \sin\theta_1 = n_a \sin\theta_2$. Expressing $\sin\theta_1$ in terms of $\sin\theta_2$, we have $\sin\theta_1 = (n_a/n_w)\sin\theta_2$. In terms of n_a, n_w, θ_2, and D_t, we can express D_a as

$$D_a = \frac{n_a D_t \cos\theta_2}{n_w \cos\theta_1} = \frac{n_a D_t \left(1 - \sin^2\theta_2\right)^{1/2}}{\left(n_w^2 - n_a^2 \sin^2\theta_2\right)^{1/2}}. \tag{1.20}$$

(2) For looking straight down at the object from a position above the water, we have $\theta_2 \to 0$. From Eq. (1.20), we see that

$$D_a\Big|_{\theta_2 \to 0} = \frac{n_a}{n_w} D_t = \frac{1.000293}{1.333} D_t \approx 0.75 D_t = \frac{3}{4} D_t. \qquad (1.21)$$

Thus, the apparent depth is about three quarters of the true depth if the object is looked straight down from a position above the water.

(3) The plot of D_a/D_t as a function of θ_2 is given in Fig. 1.8 from which we see that the apparent depth decreases monotonically as θ_2 increases. This is in consistency with our daily-life experiences.

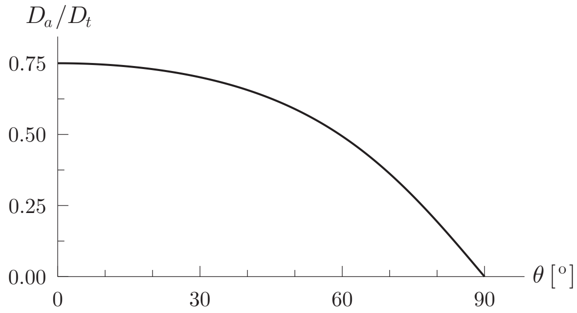

Fig. 1.8: Plot of D_a/D_t as a function of θ_2.

1.9 Refractions through a series of parallel planar interfaces. Consider the refractions of a monochromatic light beam through N parallel planar interfaces between $N+1$ layers of media numbered sequentially as shown in Fig. 1.9. The angle of incidence in the leftmost layer (layer 1) of refractive index n_1 onto the interface between layers 1 and 2 is θ_1. The angle of refraction into the rightmost layer (layer $N+1$) of refractive index n_{N+1} is θ_{N+1}. The angles of incidence and refraction in layer j $(1 < j < N)$ are equal as indicated in Fig. 1.9. Express θ_{N+1} in terms of θ_1 and refractive indices $n_1, n_2, \cdots, n_{N+1}$.

Applying the law of refraction to the refraction on the interface between layers 1 and 2, we have $n_1 \sin\theta_1 = n_2 \sin\theta_2$ from which we obtain

$$\theta_2 = \arcsin\left(\frac{n_1 \sin\theta_1}{n_2}\right).$$

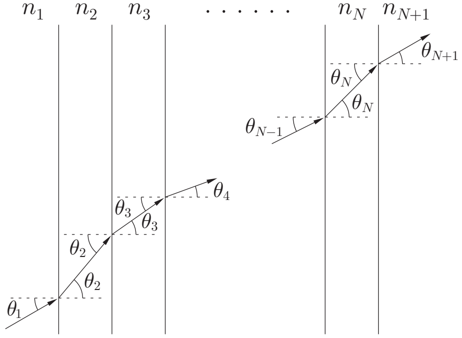

Fig. 1.9: Refractions of light through a series of parallel planar interfaces.

Applying the law of refraction to the refraction on the interface between layers 2 and 3, we have $n_2 \sin\theta_2 = n_3 \sin\theta_3$ from which we obtain

$$\theta_3 = \arcsin\left(\frac{n_2 \sin\theta_2}{n_3}\right) = \arcsin\left(\frac{n_1 n_2 \sin\theta_1}{n_2 n_3}\right).$$

Continuing in this manner until the last interface between layers N and $N+1$, we obtain

$$\theta_{N+1} = \arcsin\left(\frac{n_1 n_2 \cdots n_N \sin\theta_1}{n_2 n_3 \cdots n_{N+1}}\right) = \arcsin\left(\frac{n_1 \sin\theta_1}{n_{N+1}}\right). \quad (1.22)$$

From the above result, we see that the angle θ_{N+1} does not depend on the refractive indices of the intermediate layers between the leftmost and rightmost layers. Mathematically, this is because the factors arising from the intermediate layers are canceled completely in the successive applications of the law of refraction. Physically, this is because a light ray can be traced back and, thus, all the intermediate layers can be abstracted into an interface of zero thickness between layers 1 and $N+1$.

1.10 Concave mirror. For a concave mirror of focal length f, what is the object distance d_o such that the image distance is equal to the object distance?

Inserting $d_i = d_o$ into the mirror equation $1/d_o + 1/d_i = 1/f$, we have

$$\frac{2}{d_o} = \frac{1}{f}$$

from which it follows that

$$d_o = 2f. \tag{1.23}$$

1.11 Focal length of a concave mirror. When an object is a certain distance in front of a concave mirror, the magnification of the image is M_1. If the object is moved a distance of ℓ from its original location, the magnification of the image becomes M_2. What is the focal length of the concave mirror?

Let f denote the focal length of the concave mirror. Assume that the object was originally a distance of d_o in front of the mirror. From the mirror equation $1/d_o + 1/d_i = 1/f$, we have

$$d_i = \frac{1}{1/f - 1/d_o} = \frac{d_o f}{d_o - f}.$$

The magnification of the image is given by

$$M_1 = -\frac{d_i}{d_o} = -\frac{f}{d_o - f}. \tag{1.24}$$

For the object at $d'_o = d_o + \ell$, we have from the mirror equation

$$d'_i = \frac{1}{1/f - 1/d'_o} = \frac{f(d_o + \ell)}{d_o + \ell - f}.$$

The magnification of the image is now given by

$$M_2 = -\frac{d'_i}{d'_o} = -\frac{f}{d_o + \ell - f}. \tag{1.25}$$

Eliminating d_o from Eqs. (1.24) and (1.25), we obtain

$$f = \frac{M_2 M_1}{M_2 - M_1}\ell. \tag{1.26}$$

1.12 Observing oneself in a mirror. A person of height h from the floor to the eye level stands in front of a plane mirror with the top at the eye level of the person. What is the minimum length of the mirror for the person to be able to see in the mirror the shoes the person is wearing as much as the person can possibly see?

The condition we will use to find the minimum length of the mirror is that light reflected from the shoes can barely reach the person's eyes. Let ℓ be the length of the mirror. Let δ be the distance from the

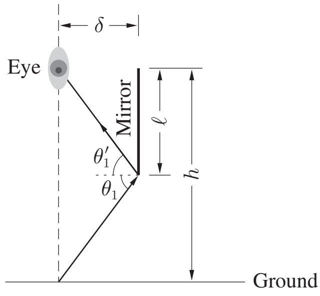

Fig. 1.10: Geometry for finding the minimum length of the mirror.

person to the mirror, with $\delta \to 0$. The relevant geometry is shown in Fig. 1.10.
The angle of incidence of light reflected from the foremost part of the bottom of a shoe onto the bottom edge of the mirror is given by

$$\tan\theta_1 = \frac{h-\ell}{\delta}.$$

The angle of reflection of light from the bottom edge of the mirror directed towards an eye of the person is given by

$$\tan\theta_1' = \frac{\ell}{\delta}.$$

From the law of reflection, we have $\theta_1 = \theta_1'$ which, for $\ell = \ell_{\rm min}$, leads to

$$\frac{h-\ell_{\rm min}}{\delta} = \frac{\ell_{\rm min}}{\delta}$$

from which we obtain $\ell_{\rm min} = h/2$. Note that this result is independent of the distance of the person to the mirror. Therefore, the minimum length of the mirror is half of the height of the person's eye level above the ground.

1.13 Focal length of a biconcave lens. A biconcave lens of refractive index n has two biconcave bounding surfaces of radius R. Find the focal length of the lens.

For a biconcave lens, we have $R_1 = -R$ and $R_2 = R$. From Lens-maker's equation $1/f = (n-1)(1/R_1 - 1/R_2)$, we have

$$\frac{1}{f} = (n-1)\left(\frac{1}{-R} - \frac{1}{R}\right) = -\frac{2(n-1)}{R}$$

The focal length of the lens is then given by

$$f = -\frac{R}{2(n-1)}. \tag{1.27}$$

From the above result, we see that the biconcave lens possesses a negave focal length and is thus a diverging lens.

1.14 Focal length of a planoconcave lens. A planoconcave lens of refractive index n has a concave bounding surface of radius R and a flat bounding surface. Find the focal length of the lens. Does the orientation of the lens affect the value of the focal length? The orientation of the lens indicates which bounding surface faces the incident light.

If the concave surface faces the incident light, we have $R_1 = -R$ and $R_2 = \infty$. From Lensmaker's equation $1/f = (n-1)(1/R_1 - 1/R_2)$, we have

$$\frac{1}{f} = (n-1)\left(\frac{1}{-R} - \frac{1}{\infty}\right) = -\frac{n-1}{R}.$$

The focal length of the lens is then given by

$$f = -\frac{R}{n-1}. \tag{1.28}$$

If the flat surface faces the incident light, we have $R_1 = \infty$ and $R_2 = R$. From Lensmaker's equation $1/f = (n-1)(1/R_1 - 1/R_2)$, we have

$$\frac{1}{f} = (n-1)\left(\frac{1}{\infty} - \frac{1}{R}\right) = -\frac{n-1}{R}$$

from which we obtain the same focal length in Eq. (1.28). Therefore, the orientation of the lens does not affect the value of the focal length.

1.15 Thin lens with a positive focal length. A thin lens of focal length $f = 0.40$ m is used to image a real object that is a distance of 0.25 m from the lens.

(1) Find the location and magnification of the image.
(2) Draw an accurate ray diagram for the image formation.

---

(1) Since the focal length of the lens is positive, the lens is a converging one. From the lens equation $1/d_o + 1/d_i = 1/f$, we have

$$d_i = \frac{1}{1/f - 1/d_o} = \frac{1}{1/0.4 - 1/0.25} \approx -0.67 \text{ m}. \tag{1.29}$$

The magnification is given by

$$M = -\frac{d_i}{d_o} = -\frac{1}{(1/f - 1/d_o)d_o} = -\frac{1}{d_o/f - 1}$$
$$= -\frac{1}{0.25/0.40 - 1} \approx 2.67. \tag{1.30}$$

That the value of M is so evaluated in the above is to avoid the rounding error. If the value of d_i in Eq. (1.29) were used in $M = -d_i/d_o$, we would have obtained $M = -d_i/d_o \approx 0.67/0.25 = 2.68$.

From the above-obtained values of d_i and M, we see that the image is virtual, upright, and magnified.

(2) The ray diagram for the image formation of the biconvex lens is given in Fig. 1.11.

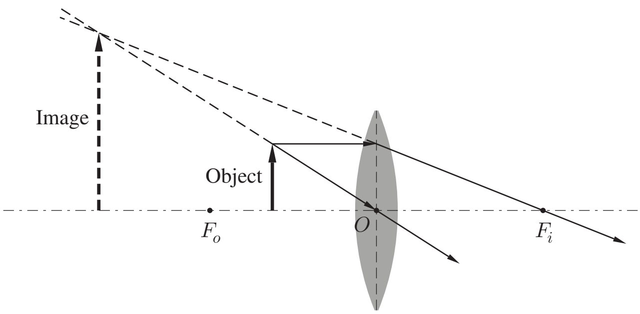

Fig. 1.11: Image formation of a biconvex lens.

1.16 Optical system with two thin lenses. An optical system [Ref. Fig. 1.12] consists of a biconvex lens (called L_1) of focal length $f_1 = 0.10$ m and a biconcave lens (called L_2) of focal length $f_2 = -0.20$ m. The distance between the two lenses is 0.10 m. Where is the image of an object in front of L_1 and a distance of 0.15 m away from it?

We first find the position of the image formed by L_1 and then find the position of the image of this image formed by L_2. From the lens equation $1/d_o + 1/d_i = 1/f$, for $d_o = 0.15$ m we have

$$d_i = \frac{d_o f_1}{d_o - f_1} = \frac{0.15 \times 0.10}{0.15 - 0.10} = 0.30 \text{ m}.$$

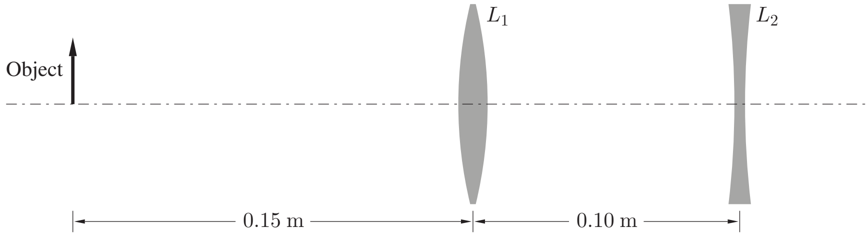

Fig. 1.12: Optical system consisting of a biconvex lens and a biconcave lens.

Thus, the image formed by L_1 is real and inverted and is 0.30 m behind L_1. Because the distance between L_1 and L_2 is 0.10 m, the image formed by L_1 is 0.20 m from L_2 on the other side of the optical system and acts as an object for L_2. Thus, $d'_o = 0.20$ m. From the lens equation $1/d_o + 1/d_i = 1/f$, we have

$$d'_i = \frac{d'_o f_2}{d'_o - f_2} = \frac{0.20 \times (-0.20)}{0.20 - (-0.20)} = -0.10 \text{ m}.$$

Thus, the image formed by the optical system is on the opposite side of the optical system with respect to the object and is 0.10 m from L_2.

1.17 Focal length of a lens. An object located a distance of 0.50 m to the right of a lens forms an image that is 0.20 m from the lens on the other side. What are the focal length f of the lens and the magnification M of the image?

From the statement of the problem, we have $d_o = 0.50$ m and $d_i = 0.20$ m. Let f be the focal length of the lens. From the lens equation $1/d_o + 1/d_i = 1/f$, we have

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{0.50} + \frac{1}{0.20} = 7 \text{ m}^{-1}$$

from which we have $f = 1/7 \approx 0.143$ m. The magnification of the image is given by

$$M = -\frac{d_i}{d_o} = -\frac{0.20}{0.50} = -0.4. \tag{1.31}$$

Thus, the image is real, inverted, and diminished.

1.18 System of converging and diverging lenses. An object is placed 0.300 m to the left of a diverging lens of focal length $f_1 = -0.100$ m.

A converging lens of focal length $f_2 = 0.300$ m is placed a distance d to the right of the diverging lens. Find the value of d so that the final image is at infinity.

We first find the image formed by the diverging lens on the left. From the lens equation $1/d_o + 1/d_i = 1/f_1$, we have

$$\frac{1}{d_i} = \frac{1}{f_1} - \frac{1}{d_o} = -\frac{1}{0.100} - \frac{1}{0.300} = -\frac{40}{3}\ \mathrm{m}^{-1}.$$

Thus, $d_i = -0.075$ m. For the image formed by the lens on the right to be at infinity, the image formed by the lens on the left must be at the focal point of the lens on the right. We thus have $f_2 = d + |d_i|$ from which we obtain

$$d = f_2 - |d_i| = 0.300 - 0.075 = 0.225\ \mathrm{m}. \tag{1.32}$$

1.19 Two identical converging lens. Two identical lens of focal length $f = 0.12$ m are placed 0.24 m apart with a common optical axis. Consider the image of an object that is placed 0.08 m to the left of the lens on the left. What is the position of the final image relative to the lens on the right?

We first consider the image formed by the lens on the left. From the lens equation $1/d_o + 1/d_i = 1/f$ for the lens on the left, we have

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o} = \frac{1}{0.12} - \frac{1}{0.08} = -\frac{25}{6}\ \mathrm{m}^{-1}$$

from which it follows that $d_i = -0.24$ m. The negative value of d_i indicates that the image is to the left of the lens on the left. Thus, the object distance to the lens on the right is given by

$$d'_o = d + |d_i| = 0.24 + 0.24 = 0.48\ \mathrm{m}.$$

Inserting the above value of d'_o to the lens equation $1/d'_o + 1/d'_i = 1/f$ for the lens on the right yields

$$\frac{1}{d'_i} = \frac{1}{f} - \frac{1}{d'_o} = \frac{1}{0.12} - \frac{1}{0.48} = 6.25\ \mathrm{m}^{-1}$$

from which we obtain $d'_i = 0.16$ m. Hence, the final image is a distance of 0.16 m to the right of the lens on the right.

1.20 Biconvex lens and convex mirror. An optical system consists of a convex lens L of focal length $f_L = 0.02$ m and a convex mirror M

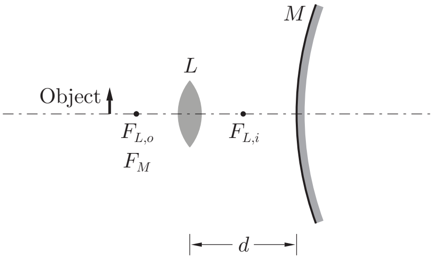

Fig. 1.13: Optical system consisting of a biconvex lens and a convex mirror. Points $F_{L,o}$ and $F_{L,i}$ are respectively the object and image focal points of the lens. Point F_M is the focal point of the mirror. Note that F_M coincides with $F_{L,o}$.

of radius $R_M = 0.12$ m as shown in Fig. 1.13. The distance between the lens and the mirror is $d = 0.04$ m. Find the location of the final image of the object located 0.03 m in front of the lens L.

We first find the image of the object formed by the lens. We then find the image of the lens-image formed by the mirror. From the lens equation $1/d_{L,o} + 1/d_{L,i} = 1/f_L$, we have

$$\frac{1}{d_{L,i}} = \frac{1}{f_L} - \frac{1}{d_{L,o}} = \frac{1}{0.02} - \frac{1}{0.03} = \frac{50}{3}\ \mathrm{m}^{-1}$$

from which we obtain $d_{L,i} = 0.06$ m. Because $d_{L,i} > d$, the image formed by the lens is a virtual object to the mirror. The object distance with respect to the mirror is then given by $d'_o = -(0.06 - 0.04) = -0.02$ m. From the mirror equation $1/d'_o + 1/d'_i = 1/f$ with $f = -R_M/2$, we have

$$\frac{1}{d'_i} = -\frac{2}{R} - \frac{1}{d'_o} = -\frac{2}{0.12} - \frac{1}{-0.02} = \frac{100}{3}\ \mathrm{m}^{-1}$$

from which it follows that $d'_i = 0.03$ m. That is, the final image is a distance of 0.03 m to the left of the mirror M.

1.21 Optical system of two biconvex thin lenses. Consider an optical system consisting of two identical biconvex thin lenses of focal length f with a separation of $2f$. For an object a distance $d_{o,1}$ in front of the front lens, what is the distance $d_{i,2}$ of the image from the back lens?

We first consider the image formed by the front lens. From the lens equation $1/d_{o,1} + 1/d_{i,1} = 1/f$ for the front lens, we have

$$\frac{1}{d_{i,1}} = \frac{1}{f} - \frac{1}{d_{o,1}} = \frac{d_{o,1} - f}{d_{o,1} f}.$$

We thus have

$$d_{i,1} = \frac{d_{o,1} f}{d_{o,1} - f}.$$

The front-lens-image acts as an object for the back lens. The object distance for the back lens is

$$d_{o,2} = 2f - d_{i,1} = \frac{f(d_{o,1} - 2f)}{d_{o,1} - f}.$$

From the lens equation $1/d_{o,2} + 1/d_{i,2} = 1/f$ for the back lens, we have

$$\frac{1}{d_{i,2}} = \frac{1}{f} - \frac{1}{d_{o,2}} = \frac{1}{f} - \frac{d_{o,1} - f}{f(d_{o,1} - 2f)} = \frac{1}{2f - d_{o,1}}$$

from which it follows that

$$d_{i,2} = 2f - d_{o,1}. \tag{1.33}$$

1.22 Parallel incident rays not parallel to the optical axis. Parallel incident rays make an angle of $\theta = 5^{\rm o}$ with the optical axis of a diverging lens of focal length $f = -0.30$ m. Where is the image?

Because all the incident rays are parallel, the virtual image will be in the focal plane. To find the location of the image, let us consider a ray that passes through the center of the lens. Since this ray does not change direction after passing through the lens, its intersection with the focal plane determines the location of the virtual image. Thus, the virtual image will be off the optical axis by the amount given by

$$\delta = |f| \tan\theta = 0.30 \times \tan 5^{\rm o} \approx 0.026 \text{ m} = 26 \text{ mm}. \tag{1.34}$$

1.23 Retro-reflecting sphere. A retro-reflecting sphere is a sphere of glass that reflects light back directly. For a sphere to be retro-reflecting, what must be its refractive index? As shown in Fig. 1.14, the back surface of the sphere is coated so that light can be reflected from the back surface. Take the refractive index of air to be unity.

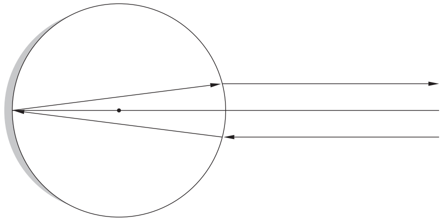

Fig. 1.14: Retro-reflecting sphere.

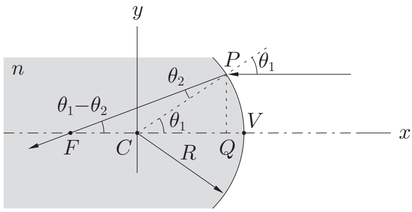

Fig. 1.15: Light ray that is parallel to the principal axis and refracts into the medium of refractive index n through a spherical surface of radius R. The focal point is denoted by F.

Let n and R be respectively the refractive index and radius of the sphere. We first consider the refraction of an incoming light ray into a medium of refractive index n through a spherical surface as shown in Fig. 1.15 and find the focal length for the spherical surface. We then apply the obtained-result to the retro-reflecting sphere.

We use the coordinate system shown in Fig. 1.15 with the coordinate origin at the center of curvature of the spherical surface. In terms of the coordinates of point P, (x_P, y_P), the angle of incidence θ_1 is given by

$$\theta_1 = \arctan\left(\frac{y_P}{x_P}\right).$$

From the law of refraction, the angle of refraction θ_2 is given by

$$\theta_2 = \arcsin\left(\frac{\sin\theta_1}{n}\right).$$

For $\theta_1 \ll 1$ and $\theta_2 \ll 1$, that is, for paraxial rays, we have

$$\tan\theta_1 \approx \sin\theta_1 \approx \theta_1 \approx \frac{y_P}{x_P},$$

$$\tan\theta_2 \approx \sin\theta_2 \approx \theta_2 \approx \frac{\theta_1}{n} \approx \frac{y_P}{nx_P}.$$

From the right triangle ΔPQF, we have

$$\overline{FQ} = \overline{PQ}\cot(\theta_1 - \theta_2) \approx \frac{\overline{PQ}}{\theta_1 - \theta_2} \approx \frac{n\overline{PQ}}{(n-1)\theta_1}.$$

From the right triangle ΔPCQ, we have

$$\overline{PQ} = R\sin\theta_1 \approx R\theta_1.$$

Thus,

$$\overline{FQ} \approx \frac{nR}{n-1}.$$

The focal length F is given by $F = \overline{FV}$. For paraxial rays, we can approximate $\overline{FV}$ with $\overline{FQ}$. The focal length f is then given by

$$f \approx \overline{FQ} \approx \frac{nR}{n-1}. \tag{1.35}$$

We now turn to the retro-reflecting sphere. For the sphere to reflect light back directly, the coated back surface of the sphere must be located at the focal point of the front spherical surface. That is, the focal length of the front spherical surface must be equal to the diameter of the sphere. We thus have

$$\frac{nR}{n-1} = 2R$$

from which it follows that

$$n = 2. \tag{1.36}$$

Note that the above result was derived under the paraxial approximation.

1.24 Three identical biconvex lenses. As shown in Fig. 1.16, three identical biconvex lenses of focal length f are aligned with two neighboring lenses separated by a distance f. Using the graphical method, find the position and magnification of the resultant image of an object located $f/2$ in front of the leftmost lens.

To construct the resultant image, we use the following two light rays: Ray 1 is incident onto the leftmost lens in parallel to the optical axis and ray 2 is directed toward the optic center of the leftmost lens. Using the rules for the image formation of thin lens, we obtain the image shown in Fig. 1.17.

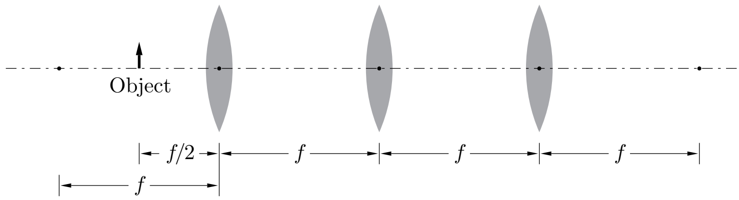

Fig. 1.16: Three aligned identical convex lenses of focal length f with a spacing of f between two neighboring lenses.

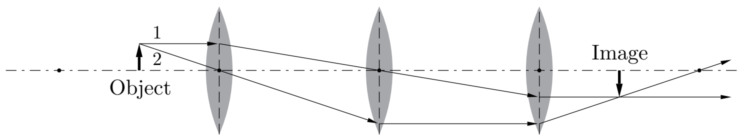

Fig. 1.17: Image formation of an optical system with three aligned identical convex lenses of focal length f with a spacing of f between two neighboring lenses.

From Fig. 1.17, we see that the image is real and inverted. Because the image is inverted, the height of the image is negative. From ray 1, we can infer that the absolute value of the height of the image is the same as the height of the object. Thus, the image is of the same size as the object.

From ray 2 and in consideration that the image is of the same size as the object, we can infer that the right triangle with the object as one side is congruent to the right triangle with the image as one side. Thus, the image is located $f/2$ to the right of the rightmost lens.

In consideration that the image is inverted and is of the same size as the object, we see that the magnification of the image is given by $M = -1$.

1.25 Determination of the object size from its image sizes. As shown in Fig. 1.18, we produced images of an object on a screen using a thin converging lens. With the positions of the object and the screen fixed, when we moved the lens in between the object and the screen, we found that sharp images on the screen could be produced with the lens being at two different positions. The heights of the sharp images are respectively given by h_1 and h_2. What is the height of the object?

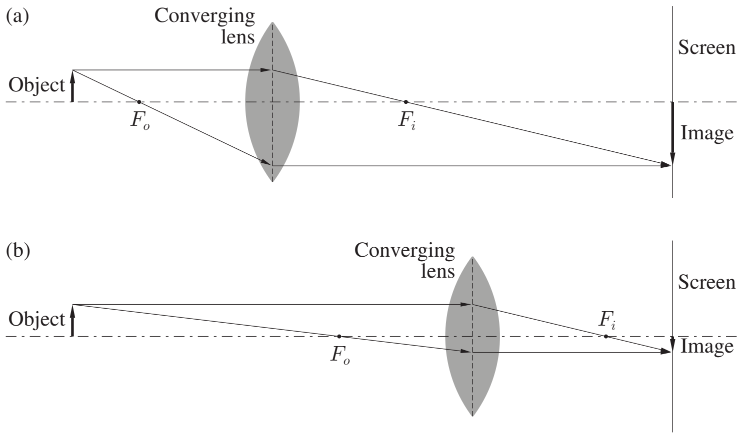

Fig. 1.18: Images of an object for the converging lens at two different positions.

Let L denote the distance between the object and the screen. Let f be the focal length of the converging lens. If the object distance is d_o, then the image distance is given by

$$d_i = L - d_o.$$

From the lens equation $1/d_o + 1/d_i = 1/f$,we have

$$\frac{1}{d_o} + \frac{1}{L - d_o} = \frac{1}{f}$$

from which we obtain the following quadratic algebraic equation for the object distance d_o

$$d_o^2 - Ld_o + fL = 0.$$

Solving for d_o from the above equation, we obtain

$$d_o = \frac{L}{2}\left(1 \pm \sqrt{1 - \frac{4f}{L}}\right). \tag{1.37}$$

The corresponding image distances are then given by

$$d_i = L - d_o = \frac{L}{2}\left(1 \mp \sqrt{1 - \frac{4f}{L}}\right). \tag{1.38}$$

From the expression $M = h_i/h_o = -d_i/d_o$ for the magnification of the image, we have

$$\frac{h_1}{h} = -\frac{1 + \sqrt{1 - 4f/L}}{1 - \sqrt{1 - 4f/L}},$$

$$\frac{h_2}{h} = -\frac{1 - \sqrt{1 - 4f/L}}{1 + \sqrt{1 - 4f/L}}.$$

Multiplying together the above two equations yields

$$\frac{h_1 h_2}{h^2} = 1.$$

Therefore,

$$h = \sqrt{h_1 h_2}. \tag{1.39}$$

1.26 Telephoto lens of a camera. As shown in Fig. 1.19, placing a diverging lens between the normal camera lens and the sensor plane transforms the camera lens into a telephoto lens. The focal lengths of the camera lens (L_1) and the diverging lens (L_2) are respectively $f_1 = 50$ mm and $f_2 = -100$ mm.

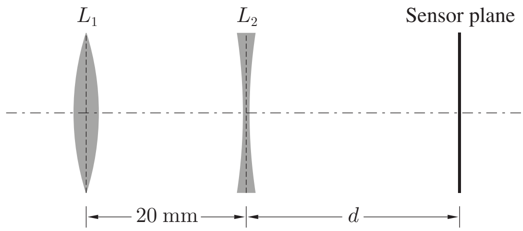

Fig. 1.19: Telephoto lens.

(1) What is the distance d between the diverging lens and the sensor plane if the lens system is focused on an object 0.5 m in front of the lens L_1?

(2) What is the magnification of an image formed by the lens system?

---

(1) With respect to L_1, the object distance is $d_{o,1} = 0.500$ m. From the lens equation $1/d_{o,1} + 1/d_{i,1} = 1/f_1$ for L_1, we have

$$\frac{1}{d_{i,1}} = \frac{1}{f_1} - \frac{1}{d_{o,1}} = \frac{1}{0.050} - \frac{1}{0.500} = 18 \text{ m}^{-1}$$

from which we have $d_{i,1} = 1/18$ m. To avoid rounding errors, we have kept the fractional value for $d_{i,1}$. The image of L_1 is the object to L_2. With respect to L_2, the object distance is $d_{o,2} = 0.020 - d_{i,1} = -16/450$ m. From the lens equation $1/d_{o,2} + 1/d_{i,2} = 1/f_2$ for L_2, we have

$$\frac{1}{d_{i,2}} = \frac{1}{f_2} - \frac{1}{d_{o,2}} = -\frac{1}{0.100} + \frac{450}{16} = \frac{290}{16} \text{ m}^{-1}$$

from which we obtain $d_{i,2} = 16/290 \approx 0.055\text{m} = 55$ mm. Thus, $d = 55$ mm in order for the image to be in the sensor plane.

(2) The magnification of the system is given by the product of the magnifications of the two lens. We have

$$M = \left(-\frac{d_{i,1}}{d_{o,1}}\right)\left(-\frac{d_{i,2}}{d_{o,2}}\right) = \frac{d_{i,1}d_{i,2}}{d_{o,1}d_{o,2}}$$
$$= -\frac{16 \times 450}{18 \times 290 \times 0.500 \times 16} \approx -0.172. \quad (1.40)$$

The negative value of M implies that the image is inverted with respect to the object.

1.27 **Simple telescope.** We consider a simple telescope consisting of three thin lens L_1, L_2, and L_3 as shown in Fig. 1.20. L_1 functions as the aperture and the entrance pupil of the telescope. L_3 is the eye lens. The focal lengths and diameters of L_1, L_2, and L_3 are respectively $f_1 = 0.100$ m, $f_2 = f_3 = 0.020$ m, $D_1 = 0.040$ m, and $D_2 = D_3 = 0.012$ m.

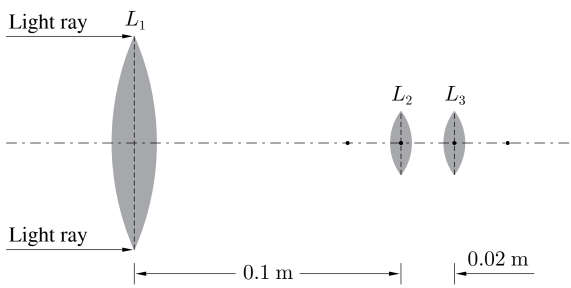

Fig. 1.20: Simple telescope.

(1) Trace a light ray entering the telescope in parallel to the optical axis through the telescope.

(2) Find the position and diameter of the exit pupil.

---

(1) Ray tracing through the system is given in Fig. 1.21 from which we see that light rays entering the system in parallel to the optical axis will be still parallel to the optical axis after passing through the system.

(2) The exit pupil is the image of the entrance pupil L_1 through L_2 and L_3. With respect to L_2, L_1 is located at a distance of 0.100 m away so that the object distance is given by

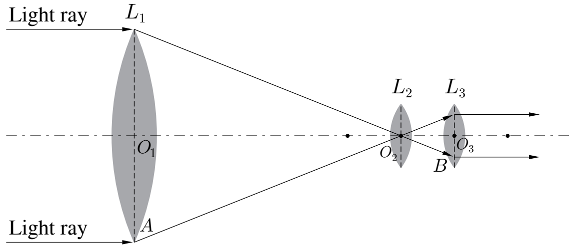

Fig. 1.21: Ray tracing through the simple telescope.

$d_{o,2} = 0.100$ m. From the lens equation $1/d_{o,2} + 1/d_{i,2} = 1/f_2$ for L_2, we have

$$\frac{1}{d_{i,2}} = \frac{1}{f_2} - \frac{1}{d_{o,2}} = \frac{1}{0.020} - \frac{1}{0.100} = 40 \text{ m}^{-1}$$

from which it follows that $d_{i,2} = 0.025$ m. The image formed by L_2 is the object to L_3. In consideration that the distance between L_2 and L_3 is 0.020 m, we have

$$d_{o,3} = 0.020 - 0.025 = -0.005 \text{ m}.$$

From the lens equation $1/d_{o,3} + 1/d_{i,3} = 1/f_3$ for L_3, we have

$$\frac{1}{d_{i,3}} = \frac{1}{f_3} - \frac{1}{d_{o,3}} = \frac{1}{0.020} + \frac{1}{0.005} = 250 \text{ m}^{-1}$$

from which we have $d_{i,3} = 0.004$ m. Thus, the exit pupil is located 0.004 m behind L_3.
The diameter of the exit pupil can be found by using the similarity between the right triangles formed by light rays, optical axis, and lenses. The diameter of the entrance pupil is given by the diameter D_1 of L_1. Let D' denote the diameter of the exit pupil. From the similarity between the right triangles ΔAO_1O_2 and ΔBO_3O_2, we have

$$\frac{D_1/2}{D'/2} = \frac{f_1}{f_3}.$$

Thus,

$$D' = \frac{f_3}{f_1} D_1 = \frac{0.020}{0.100} \times 0.040 = 0.008 \text{ m}. \qquad (1.41)$$

1.28 Prescription for a farsighted eye. The eye of a farsighted person has a near point at 1.25 m. What prescription in diopters will be required for contact lenses if the near point of the eye of that person is to be moved to 0.25 m?

For contact lenses, the optical power of the corrective lenses is given by $1/f = 1/d_{\text{near,normal}} - 1/d_{\text{near}}$. In the current case, $d_{\text{near}} = 1.25$ m and $d_{\text{near,normal}} = 0.25$ m. We thus have

$$\frac{1}{f} = \frac{1}{d_{\text{near,normal}}} - \frac{1}{d_{\text{near}}} = \frac{1}{0.25} - \frac{1}{1.25} = 3.2 \text{ diopters.} \qquad (1.42)$$

1.29 ***Focusing through a refracting interface.** Parallel light rays are incident from medium 1 of refractive index n_1 onto the interface between medium 1 and medium 2 of refractive index n_2 as shown in Fig. 1.22. Assume that $n_2 > n_1$.

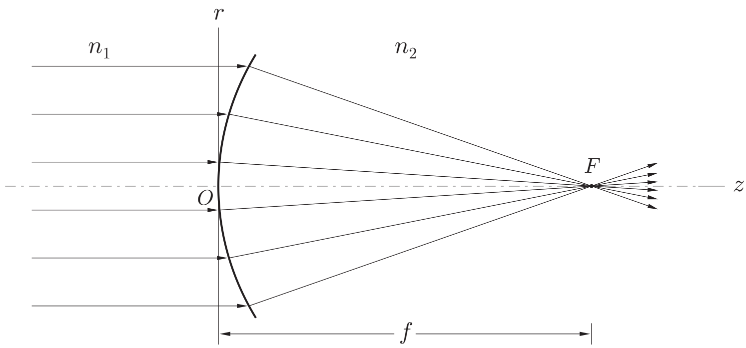

Fig. 1.22: Focusing interface between two media.

(1) For the light rays to be focused at the point F that is a distance f from point O, what is the shape of the interface?

(2) Find the maximum radius of the light beam that can be focused by the interface.

---

(1) Let us consider the light ray that is incident at point (z, r) on the interface as shown in Fig. 1.23.
The angle of incidence is given by

$$\tan\theta_1 = \frac{dz}{dr}.$$

From the law of refraction, the angle of refraction is given by

$$\sin\theta_2 = \frac{n_1 \sin\theta_1}{n_2}.$$

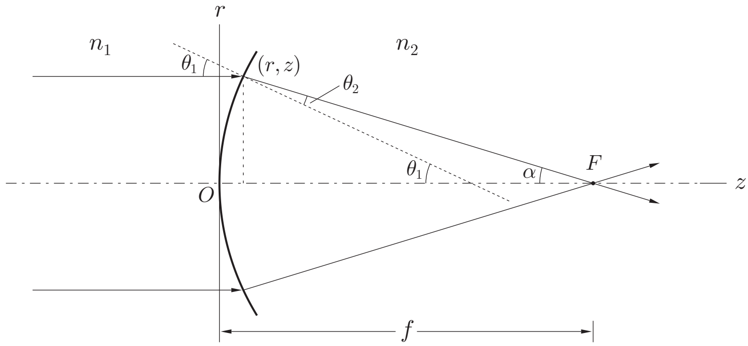

Fig. 1.23: Refraction of a light ray by the interface between two media.

For the light ray to be focused at F, we must have $\theta_2 = \theta_1 - \alpha$ with

$$\tan\alpha = \frac{r}{f-z},\ \sin\alpha = \frac{r}{\sqrt{r^2+(f-z)^2}},\ \cos\alpha = \frac{f-z}{\sqrt{r^2+(f-z)^2}}.$$

Taking the sine of both sides of the relation $\theta_2 = \theta_1 - \alpha$, we have

$$\sin\theta_2 = \sin(\theta_1 - \alpha),$$

$$\frac{n_1 \sin\theta_1}{n_2} = \cos\alpha \sin\theta_1 - \sin\alpha\cos\theta_1,$$

$$\tan\theta_1 = -\frac{\sin\alpha}{n_1/n_2 - \cos\alpha},$$

$$\frac{dz}{dr} = -\frac{r}{(n_1/n_2)\sqrt{r^2+(f-z)^2} - (f-z)}. \tag{1.43}$$

To simplify the above equation, we introduce two new variables η and ρ that are related to z and r through

$$\eta = f - z, \quad \rho = \sqrt{r^2+(f-z)^2}.$$

We then have

$$\frac{d}{dz} = -\frac{d}{d\eta},$$

$$\frac{d\rho}{d\eta} = -\frac{d}{dz}\sqrt{r^2+(f-z)^2} = -\frac{1}{\rho}\left(r\frac{dr}{dz} - \eta\right).$$

Plugging the expression for rdr/dz from Eq. (1.43) into the above equation, we have

$$\frac{d\rho}{d\eta} = \frac{n_1}{n_2}.$$

Integrating the above equation under the boundary condition that $\rho|_{\eta=f} = f$, we have

$$\rho - f = \frac{n_1}{n_2}(\eta - f).$$

Inserting the expressions of η and ρ in terms of r and z into the above equation, we have

$$\sqrt{r^2 + (f-z)^2} - f = -\frac{n_1}{n_2} z.$$

Solving for z from the above equation, we obtain

$$z = \frac{n_2 f}{n_2 + n_1}\left[1 \pm \sqrt{1 - \frac{(n_2+n_1)r^2}{(n_2-n_1)f^2}}\right].$$

The result with the plus sign in the above equation is not consistent with the boundary condition $z|_{r=0} = 0$. Thus, the shape of the interface is given by

$$z = \frac{n_2 f}{n_2 + n_1}\left[1 - \sqrt{1 - \frac{(n_2+n_1)r^2}{(n_2-n_1)f^2}}\right]. \qquad (1.44)$$

(2) The presence of the square root in Eq. (1.44) indicates that the values of r can not be larger than a certain value $r_{\max}$ that can be obtained by setting the quantity under the square root to zero. We then have

$$1 - \frac{n_2+n_1}{n_2-n_1}\frac{r_{\max}^2}{f^2} = 0$$

from which we have

$$r_{\max} = \sqrt{\frac{n_2-n_1}{n_2+n_1}}\, f. \qquad (1.45)$$

1.30 ***Spherical drop of water.** Consider a light ray enters and leaves a spherical drop of water of index n as shown in Fig. 1.24.

(1) What is the angle of incidence β on the back surface?

(2) What is the angle of deflection γ when the ray exits the water drop?

(3) What is the angle of incidence α that yields the smallest deflection?

---

(1) From Fig. 1.24, we see that the angle of incidence β on the back surface is equal to the angle of refraction at point A (the entrance

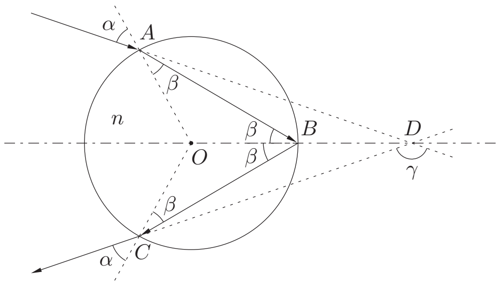

Fig. 1.24: Reflection and refractions of a light ray on a spherical drop of water. The equal angles have been identified.

point of the light ray into the water drop). Applying the law of refraction to the refraction at A, we have $\sin\alpha = n\sin\beta$ from which it follows that

$$\beta = \arcsin\left(\frac{\sin\alpha}{n}\right). \tag{1.46}$$

(2) From triangles ΔABD and ΔCBD, we have

$$\begin{aligned}\gamma &= \pi - 2\big[\beta - (\alpha - \beta)\big] = \pi + 2\alpha - 4\beta \\ &= \pi + 2\alpha - 4\arcsin\left(\frac{\sin\alpha}{n}\right). \end{aligned} \tag{1.47}$$

(3) Differentiating γ with respect to α and setting the result to zero, we obtain

$$\frac{d\gamma}{d\alpha} = 2 - 4\frac{\cos\alpha}{\sqrt{n^2 - \sin^2\alpha}} = 0$$

from which we obtain

$$\begin{aligned}\cos^2\alpha &= \frac{n^2-1}{3}, \\ \alpha &= \arccos\sqrt{\frac{n^2-1}{3}}. \end{aligned} \tag{1.48}$$

1.31 **Deflection of a light ray passing through a prism. Consider a light ray passing through an isosceles prism of refractive index n and refracting angle α. For the refracting angle α of the prism and other angles, see Fig. 1.25. The deflection angle of the light ray is denoted by γ. The refractive index of air is taken to be unity.

(1) Express γ in terms of α, θ_1, and θ_2'.

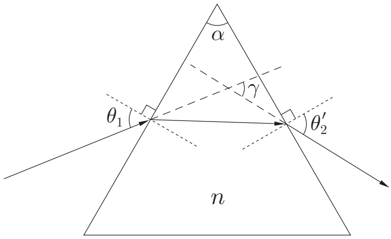

Fig. 1.25: Prism and light ray passing through it.

(2) Find a relation between γ and α for a minimum value of γ.

(3) If the light ray passes through the prism symmetrically, show that the relation derived in **(2)** is satisfied.

(4) Show that $\gamma \approx (n-1)\alpha$ if both α and θ_1 are small.

---

(1) With the angles of incidence and refraction at the front and back surfaces of the prism shown, the prism and the light ray passing through it are given again in Fig. 1.26. For the convenience of showing the angles more clearly, we have used a large angle of incidence in Fig. 1.26 which is exact in the sense that the path of the light ray is constructed exactly in accordance with the law of refraction.

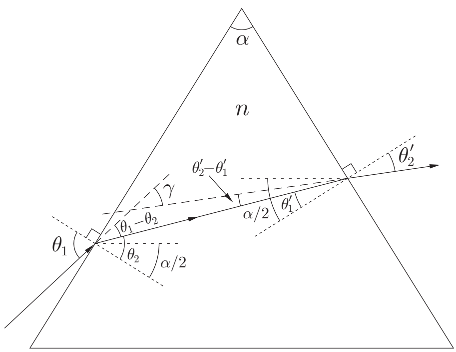

Fig. 1.26: Prism and light ray passing through it with all the angles of incidence and refraction shown.

From Fig. 1.26, we see that γ can be expressed in terms of angles θ_1, θ_2, θ_1', and θ_2' as follows

$$\gamma = \theta_1 - \theta_2 + \theta_2' - \theta_1'.$$

From Fig. 1.26, we can infer the following relation between angles θ_1' and θ_2

$$\theta_1' + \theta_2 = \alpha. \tag{1.49}$$

We thus have

$$\gamma = \theta_1 - \theta_2 + \theta_2' - (\alpha - \theta_2) = \theta_1 + \theta_2' - \alpha. \tag{1.50}$$

(2) To minimize γ, we differentiate γ with respect to θ_1 and set the result to zero. We have

$$0 = \frac{d\gamma}{d\theta_1} = 1 + \frac{d\theta_2'}{d\theta_1}. \tag{1.51}$$

To evaluate the right hand side of the above equation, we write down the law of refraction for refractions at the front and back surfaces of the prism and have

$$\sin\theta_1 = n\sin\theta_2, \tag{1.52}$$

$$n\sin\theta_1' = \sin\theta_2'. \tag{1.53}$$

Making use of Eqs. (1.53), (1.49), and (1.52) in this order, we have

$$\frac{d\theta_2'}{d\theta_1} = n\frac{\cos\theta_1'}{\cos\theta_2'}\frac{d\theta_1'}{d\theta_1} = -n\frac{\cos\theta_1'}{\cos\theta_2'}\frac{d\theta_2}{d\theta_1} = -\frac{\cos\theta_1'\cos\theta_1}{\cos\theta_2'\cos\theta_2}.$$

Inserting the above result into Eq. (1.51) yields

$$\cos\theta_1\cos\theta_1' = \cos\theta_2\cos\theta_2'.$$

Multiplying Eq. (1.52) with Eq. (1.53), we have

$$\sin\theta_1\sin\theta_1' = \sin\theta_2\sin\theta_2'.$$

Adding up the above two equations yields

$$\cos(\theta_2 - \theta_2') - \cos(\theta_1 - \theta_1') = 0.$$

Making use of the identity $\cos\alpha - \cos\beta = -2\sin[(\alpha + \beta)/2]\sin[(\alpha - \beta)/2]$ from elementary trigonometry, we have

$$\sin\frac{\theta_2 - \theta_2' + \theta_1 - \theta_1'}{2}\sin\frac{\theta_2 - \theta_2' - \theta_1 + \theta_1'}{2} = 0.$$

Inserting Eq. (1.49) into the second factor on the left hand side of the above equation yields

$$\sin\frac{\theta_2 - \theta_2' + \theta_1 - \theta_1'}{2}\sin\frac{\theta_1 + \theta_2' - \alpha}{2} = 0.$$

Making use of Eq. (1.50), we obtain

$$\sin\frac{\theta_2-\theta_2'+\theta_1-\theta_1'}{2}\sin\frac{\gamma}{2}=0.$$

Because γ can not be zero or 2π, we must have $\theta_2-\theta_2'+\theta_1-\theta_1'=0$, that is,

$$\theta_1-\theta_2'=\theta_1'-\theta_2. \tag{1.54}$$

To derive the desired relation between γ and α, we again make use of Eqs. (1.52) and (1.53). Interchanging the two sides of Eq. (1.53) and then adding the resultant equation to Eq. (1.52), we obtain

$$\sin\theta_1+\sin\theta_2'=n\left(\sin\theta_2+\sin\theta_1'\right).$$

Making use of the identity $\sin\alpha+\sin\beta=2\sin[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$, we have

$$\sin\frac{\theta_1+\theta_2'}{2}\cos\frac{\theta_1-\theta_2'}{2}=n\sin\frac{\theta_2+\theta_1'}{2}\cos\frac{\theta_2-\theta_1'}{2}.$$

Due to Eq. (1.54), the second factor on the left hand side cancels the second factor on the right hand side in the above equation, which leads to

$$\sin\frac{\theta_1+\theta_2'}{2}=n\sin\frac{\theta_2+\theta_1'}{2}.$$

Making use of Eqs. (1.49) and (1.50), we obtain

$$\sin\frac{\gamma+\alpha}{2}=n\sin\frac{\alpha}{2} \tag{1.55}$$

which is the desired relation.

(3) If the light ray passes through the prism symmetrically, we have

$$\theta_1=\theta_2',\quad \theta_2=\theta_1'.$$

Because of Eq. (1.49), we have

$$\theta_2=\theta_1'=\frac{\alpha}{2}.$$

The deflection angle is now given by

$$\gamma=\theta_1+\theta_2'-\alpha=2\theta_1-\alpha.$$

We then have

$$\sin\frac{\gamma+\alpha}{2}=\sin\theta_1=n\sin\theta_2=n\sin\frac{\alpha}{2}.$$

Therefore, the relation in Eq. (1.55) is satisfied if the light ray passes through the prism symmetrically.

(4) If both α and θ_1 are small, then θ_2, θ_1', and θ_2' are also small. In this case, Eqs. (1.52) and (1.53) become

$$\theta_1 \approx n\theta_2, \;\; n\theta_1' \approx \theta_2'.$$

Making use of the above two equations and Eq. (1.49), $\theta_1' + \theta_2 = \alpha$, we have

$$\begin{aligned}\gamma &= \theta_1 + \theta_2' - \alpha \approx n\theta_2 + n\theta_1' - \alpha \\ &= n(\theta_2 + \theta_1') - \alpha = n\alpha - \alpha = (n-1)\alpha. \qquad (1.56)\end{aligned}$$

1.32 **Thin biconcave lens with one surface silvered. A convex spherical mirror is obtained by coating with silver one of the two surfaces of a thin biconcave lens whose refractive index is n and whose two surfaces have the same radius of curvature R. What is the focal length of the resultant convex spherical mirror?

For clarity, we illustrate the geometry in Fig. 1.27 for a thick biconcave lens of thickness t. However, *our calculations will be performed for a thin biconcave lens with* $t \to 0$. We follow an incident light ray parallel to the optical axis and find the point where the backward extension of the light ray intersects the optical axis when it emerges from the lens. We will make use of the paraxial approximation from the beginning of our calculations.

The light ray enters the lens at point P, is reflected at point Q on the back surface, and emerges from the lens at point S on the front surface. The backward extension of the emerging ray intersects the optical axis at point F that is the focal point of the convex spherical mirror. Because point F is to the right of point O, the focal length f is negative and is given by $f = -\overline{OF}$. Because the focal length is negative, the resultant mirror is a *convex* spherical mirror as we have been referring to.

Assume that the incident ray is a distance of h above the optical axis with $h \ll R$. The limit of $h \to 0$ will be taken at the end of our calculations. The coordinates of point P in the paraxial approximation are given by

$$\begin{aligned}x_P &= \sqrt{R^2 - h^2} - R \approx -\frac{h^2}{2R}, \\ y_P &= h.\end{aligned}$$

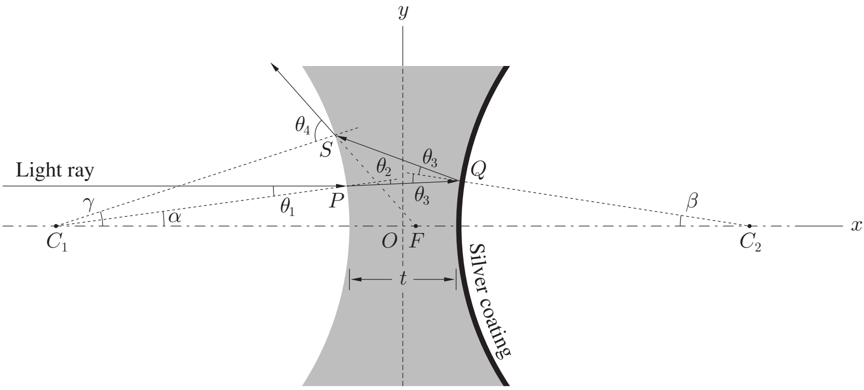

Fig. 1.27: Determination of the focal length of a biconvex lens with one surface coated with silver.

The value of angle α in the paraxial approximation is given by

$$\alpha = \arctan\frac{y_P}{R-|x_P|} \approx \frac{y_P}{R} \approx \frac{h}{R}.$$

The angle of incidence θ_1 at point P is equal to α, $\theta_1 = \alpha$. From the law of refraction, we have

$$\theta_2 \approx \sin\theta_2 = \frac{\sin\theta_1}{n} \approx \frac{\theta_1}{n} \approx \frac{h}{nR}.$$

We now examine the reflection at point Q. First, we find the coordinates of point Q. The path of the light ray from P to Q is described by $y = \tan(\theta_1 - \theta_2)(x - x_P) + y_P$ and the back surface of the lens is described by $(x-R)^2 + y^2 = R^2$. Solving for x and y from these two equations with the application of the paraxial approximation yields

$$x_Q \approx \frac{h^2}{2R},$$
$$y_Q \approx h.$$

The angle β is then given by

$$\beta \approx \tan\beta = \frac{y_Q}{R - x_Q} \approx \frac{y_Q}{R} \approx \frac{h}{R}.$$

The angle of incidence θ_3 at point Q is

$$\theta_3 = \beta + \theta_1 - \theta_2 \approx \frac{(2n-1)h}{nR}.$$

We next examine the refraction at point S. Again, we first find the coordinates of point S. The path of the light ray from Q to

S is described by $y = -\tan(\beta + \theta_3)(x - x_Q) + y_Q$ and the front surface of the lens is described by $(x+R)^2 + y^2 = R^2$. Solving for x and y from these two equations with the application of the paraxial approximation yields

$$x_S \approx -\frac{(n+1)h^2}{2nR},$$
$$y_S \approx h.$$

From the above results, we obtain the following value for angle γ

$$\gamma = \arctan\frac{y_S}{R - |x_S|} \approx \frac{y_S}{R} \approx \frac{h}{R}.$$

The angle of incidence at point S is given by

$$\theta_3 + \beta + \gamma \approx \frac{(2n-1)h}{nR} + \frac{2h}{R} = \frac{(4n-1)h}{nR}.$$

From the law of refraction, the angle of refraction θ_4 at point S is given by

$$\theta_4 \approx \sin\theta_4 = n\sin(\theta_3 + \beta + \gamma) \approx n(\theta_3 + \beta + \gamma) \approx \frac{(4n-1)h}{R}.$$

The slope of the outgoing ray is then given by

$$k_o = -\tan(\theta_4 - \gamma) \approx -(\theta_4 - \gamma) \approx -\frac{2(2n-1)h}{R}.$$

The equation for the outgoing ray reads

$$y = k_o(x - x_S) + y_S.$$

The x-coordinate of the focal point F can be obtained by setting $y = 0$ in the above equation. We have

$$x_F = -\frac{y_S}{k_o} + x_S \approx \frac{R}{2(2n-1)} - \frac{(n+1)h^2}{2nR} \approx \frac{R}{2(2n-1)}.$$

In the limit of $h \to 0$, the above result for x_F becomes exact. Hence, the focal length of the resultant convex spherical mirror is given by

$$f = -\overline{OF} = -x_F = -\frac{R}{2(2n-1)}. \qquad (1.57)$$

1.33 **Thin biconvex lens with one surface silvered. A concave spherical mirror is obtained by coating with silver one of the two surfaces of a thin biconvex lens whose refractive index is n and whose two surfaces have the same radius of curvature R. What is the focal length of the resultant concave spherical mirror?

To find the focal point and the focal length of the resultant concave spherical mirror, we follow an incident light ray parallel to the optical axis and find the point where the light ray intersects the optical axis when it emerges from the lens. The relevant geometry is illustrated in Fig. 1.28 for a thick biconconvex lens of thickness t. We will apply the paraxial approximation from the beginning of our calculations.

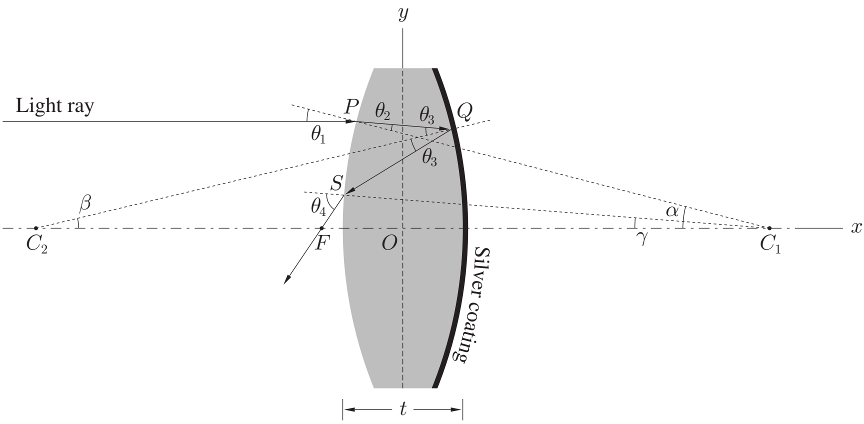

Fig. 1.28: Determination of the focal length of a biconvex lens with one surface coated with silver.

The light ray enters the lens at point P, is reflected at point Q on the back surface, and emerges from the lens at point S on the front surface. The emerging ray intersects the optical axis at point F that is the focal point of the concave spherical mirror. Because point F is to the left of point O, the focal length f is positive and is given by $f = \overline{FO}$. Because the focal length is positive, the resultant mirror is a *concave* spherical mirror as we have been referring to.

Assume that the incident ray is a distance of h above the optical axis with $h \ll R$. The limit of $h \to 0$ and $t \to 0$ will be taken at the end of our calculations. The coordinates of point P in the paraxial approximation are given by

$$x_P = -\sqrt{R^2 - h^2} + R - \frac{t}{2} \approx \frac{h^2}{2R} - \frac{t}{2},$$
$$y_P = h.$$

Because the h-dependence of x_P is of the second order in h, $x_P \approx -t/2$ up to the first order in both t and h.

The value of angle α in the paraxial approximation is given by

$$\alpha = \arctan \frac{y_P}{R - (t/2 - |x_P|)} \approx \frac{y_P}{R - h^2/2R} \approx \frac{h}{R}.$$

The angle of incidence θ_1 at point P is equal to α, $\theta_1 = \alpha$. From the law of refraction, we have

$$\theta_2 \approx \sin\theta_2 = \frac{\sin\theta_1}{n} \approx \frac{\theta_1}{n} \approx \frac{h}{nR}.$$

We now examine the reflection at point Q. First, we find the coordinates of point Q. The path of the light ray from P to Q is described by $y = -\tan(\theta_1 - \theta_2)(x - x_P) + y_P$ and the back surface of the lens is described by $(x + R - t/2)^2 + y^2 = R^2$. Solving for x and y from these two equations with the application of the paraxial approximation yields

$$x_Q \approx \frac{t}{2} - \frac{h^2}{2R},$$
$$y_Q \approx h.$$

Because the h-dependence of x_Q is of the second order in h, $x_Q \approx t/2$ up to the first order in both t and h.

The angle β is then given by

$$\beta \approx \tan\beta = \frac{y_Q}{R - (t/2 - x_Q)} \approx \frac{y_Q}{R} \approx \frac{h}{R}.$$

The angle of incidence θ_3 at point Q is

$$\theta_3 = \beta + \theta_1 - \theta_2 \approx \frac{(2n-1)h}{nR}.$$

We next examine the refraction at point S. Again, we first find the coordinates of point S. The path of the light ray from Q to S is described by $y = \tan(\beta + \theta_3)(x - x_Q) + y_Q$ and the front surface of the lens is described by $(x - R + t/2)^2 + y^2 = R^2$. Solving for x

and y from these two equations with the application of the paraxial approximation yields

$$x_S \approx -\frac{t}{2} + \frac{(n+1)h^2}{2nR},$$
$$y_S \approx h.$$

Because the h-dependence of x_S is of the second order in h, $x_S \approx -t/2$ up to the first order in both t and h. From the above results, we obtain the following value for angle γ

$$\gamma = \arctan \frac{y_S}{R - (t/2 - |x_S|)} \approx \frac{y_S}{R} \approx \frac{h}{R}.$$

The angle of incidence at point S is given by

$$\theta_3 + \beta + \gamma \approx \frac{(2n-1)h}{nR} + \frac{2h}{R} = \frac{(4n-1)h}{nR}.$$

From the law of refraction, the angle of refraction θ_4 at point S is given by

$$\theta_4 \approx \sin\theta_4 = n\sin(\theta_3 + \beta + \gamma) \approx n(\theta_3 + \beta + \gamma) \approx \frac{(4n-1)h}{R}.$$

The slope of the outgoing ray is then given by

$$k_o = \tan(\theta_4 - \gamma) \approx \theta_4 - \gamma \approx \frac{2(2n-1)h}{R}.$$

The equation for the outgoing ray reads

$$y = k_o(x - x_S) + y_S.$$

The x-coordinate of the focal point F can be obtained by setting $y = 0$ in the above equation. We have

$$x_F = -\frac{y_S}{k_o} + x_S \approx -\frac{R}{2(2n-1)} + \frac{t}{2} - \frac{(n+1)h^2}{2nR} \to -\frac{R}{2(2n-1)},$$

where we have taken the limit of $t \to 0$ and $h \to 0$. Therefore, the focal length of the resultant convex spherical mirror is given by

$$f = \overline{FO} = -x_F = \frac{R}{2(2n-1)}. \qquad (1.58)$$

1.34 **Ray-tracing calculation for a planoconvex lens. The geometrical spot size produced by a collimated beam incident on a planoconvex lens can be estimated through an exact ray-tracing calculation. Assume that the refractive index, radius of curvature, diameter, and center thickness of the lens are respectively $n = 1.5$,

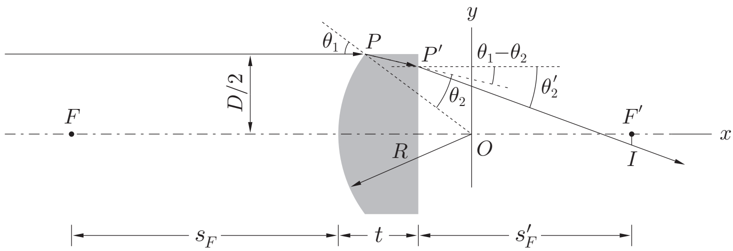

Fig. 1.29: Planoconvex lens and path of a light ray through the lens. For clarity, the thickness of the lens has been exaggerated.

$R = 0.050$ m, $D = 0.060$ m, and $t = 0.015$ m. The refractive index of air is taken to be unity. Consider the incoming ray parallel to the optical axis and incident at the edge of the lens as shown in Fig. 1.29.

(1) Find the coordinates of the entrance point into the lens and the angles of incidence and refraction at the front surface.

(2) Find the angles of incidence and refraction at the back surface and the coordinates of the exit point from the lens.

(3) Find the effective focal length f, the front focal length s_F, and the back focal length $s_{F'}$.

(4) Find the height $h = \overline{F'I}$ of the ray in the paraxial focal plane. The height h can be used as an estimate of the geometrical spot size.

---

(1) We use the coordinate system given in Fig. 1.29. The y-coordinate of the entrance point P is obviously $y_P = D/2 = 0.030$ m with D the given diameter of the lens, $D = 0.060$ m. The x-coordinate of the entrance point P is then given by

$$x_P = -\sqrt{R^2 - y_P^2} = -\sqrt{R^2 - D^2/4} = -0.04 \text{ m}. \tag{1.59}$$

The angle of incident at P is given by

$$\theta_1 = \arctan\left(\frac{y_P}{|x_P|}\right) \approx 36.870^{\text{o}}. \tag{1.60}$$

From the law of refraction, the angle of refraction at P is given by

$$\theta_2 = \arcsin\left(\frac{\sin\theta_1}{n}\right) \approx 23.578^{\text{o}}. \tag{1.61}$$

(2) From the geometry in Fig. 1.29, we see that the angle of incidence at P' is given by

$$\theta_1' = \theta_1 - \theta_2 \approx 13.292^{\text{o}}. \tag{1.62}$$

From the law of refraction, the angle of refraction at P' is given by

$$\theta_2' = \arcsin\left(n\sin\theta_1'\right) \approx 20.174^\circ. \tag{1.63}$$

The thickness of the lens at P is

$$t_P = R - |x_P| = R + x_P = 0.010\text{ m}.$$

Thus, the coordinates of P' are given by

$$x_{P'} = x_P + t_P = -0.030\text{ m}, \tag{1.64a}$$

$$y_{P'} = y_P - t_P\tan(\theta_1 - \theta_2) \approx 0.028\text{ m}. \tag{1.64b}$$

(3) The effective focal length can be obtained from Lensmaker's equation. For this planoconvex lens, $R_1 = R = 0.05$ m, $R_2 = \infty$, $t = 0.015$ m, and $n = 1.5$. From Lensmaker's equation, we have

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)t}{nR_1R_2}\right]$$
$$= (1.5-1)\times\left[\frac{1}{0.05} - 0 + 0\right] = 10\text{ m}^{-1}$$

from which we obtain

$$f = 0.1\text{ m}. \tag{1.65}$$

The front focal length is given by

$$s_F = f\left[1 + \frac{(n-1)t}{nR_2}\right] = 0.1\times[1+0] = 0.1\text{ m}. \tag{1.66}$$

Thus, the front focal length is equal to the effective focal length. The back focal length is given by

$$s_{F'} = f\left[1 - \frac{(n-1)t}{nR_1}\right]$$
$$= 0.1\times\left[1 - \frac{(1.5-1)\times 0.015}{1.5\times 0.05}\right] = 0.09\text{ m}. \tag{1.67}$$

(4) To find the value of $h = \overline{F'I}$, we first find the y-coordinate of point I. From Fig. 1.29, we have

$$y_I = y_{P'} - s_{F'}\tan\theta_2' \approx 0.028 - 0.09\times\tan 20.174^\circ \approx -0.005\text{ m}.$$

Thus,

$$h = \overline{F'I} = -y_I \approx 0.005\text{ m}. \tag{1.68}$$

In order to see if large rounding errors have been introduced into the above result, we insert all the previously-obtained analytical results into the expression of y_I less the expression of $s_{F'}$ since the numerical value from its expression gives the position of the desired paraxial focal plane. We have

$$\begin{aligned}
h &= s_{F'} \tan\theta_2' - y_{P'} \\
&= s_{F'} \tan\{\arcsin[\,n\sin(\theta_1-\theta_2)\,]\} - y_P + t_P\tan(\theta_1-\theta_2) \\
&= -y_P + \frac{n s_{F'}\sin(\theta_1-\theta_2)}{\sqrt{1-n^2\sin^2(\theta_1-\theta_2)}} + t_P\tan(\theta_1-\theta_2) \\
&= -y_P + \frac{n s_{F'}\sin[\arcsin(D/2R)-\arcsin(D/2nR)]}{\sqrt{1-n^2\sin^2[\arcsin(D/2R)-\arcsin(D/2nR)]}} \\
&\quad + t_P\tan\left(\arcsin\frac{D}{2R}-\arcsin\frac{D}{2nR}\right). \qquad (1.69)
\end{aligned}$$

In the above expression of h, the previously-given values of quantities y_P and t_P are exact. Evaluating h from the above expression, we obtain $h \approx 0.00543$ m which is in agreement with the previously-obtained value.