Excitability in optics

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Outline

Excitable systems: excitability, relaxation oscillations and noise effects
Thermo-optical dynamics in semiconductor optical amplifiers

High-dimensional excitable systems: chaotic spiking and excitability of non-trivial attractors (limit cycles, chaotic attractors)
Optomechanical-resonators
Semiconductor lasers with optoelectronic feedback

Spatially extended excitable systems: excitable waves
Broad-area vertical cavity optical amplifiers

Excitability

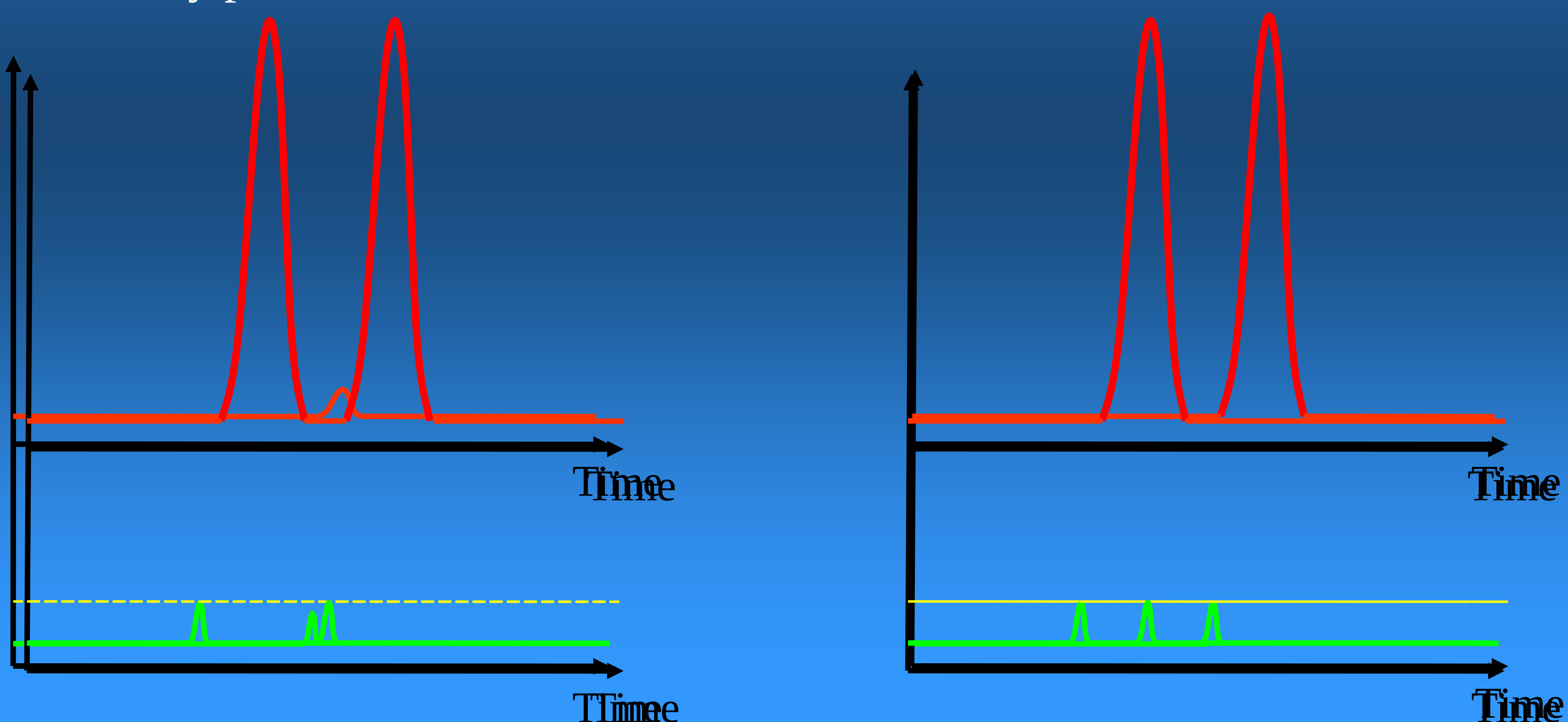
Phenomenological definition

An excitable system displays a threshold-like response to an external perturbation:

1) Below the critical value the system responds proportionally to the perturbation, and returns to its quiescent state.

2) Above the critical value the system responds independently (*in amplitude and duration*) of the size of the perturbation, before returning to the quiescent state.

3) Refractory period



Excitability

Dynamical mechanism

Minimal ingredients for an excitable dynamical system

a) Stable fixed point
b) Threshold
c) Mechanism able to re-inject trajectories crossing the threshold in the vicinity of the fixed point

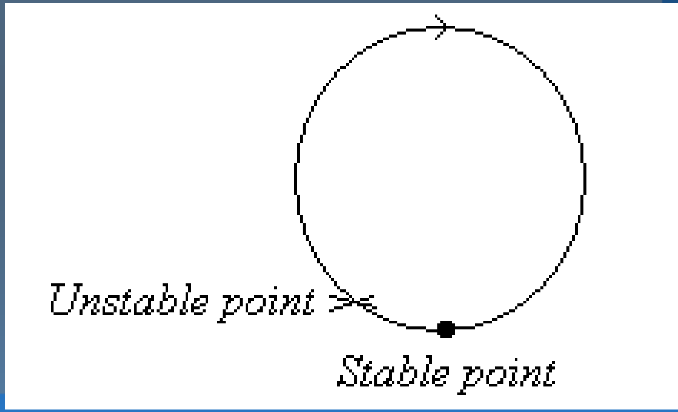
A simple way to account for this re-injection is through the topology of the phase space: Adler equation (in S^1)

$$\dot{x} = \mu - \cos x, \quad |\mu| < 1$$

x: Angular variable (modulo 2π)

(Fixed points)

$$x_{\pm} = \pm \arccos \mu$$



If we perturb the system with a perturbation larger than the distance between the fixed points, the trajectory will evolve returning to the initial state after completing one full turn to the circle (in a time T_r) $\Rightarrow$ Excitable pulse

Excitability

Threshold implies nonlinearity

$$d_t x = f(x)$$
$$x(0) = x_s$$

$$I(t) = p_0\,\delta(t - t_0)$$

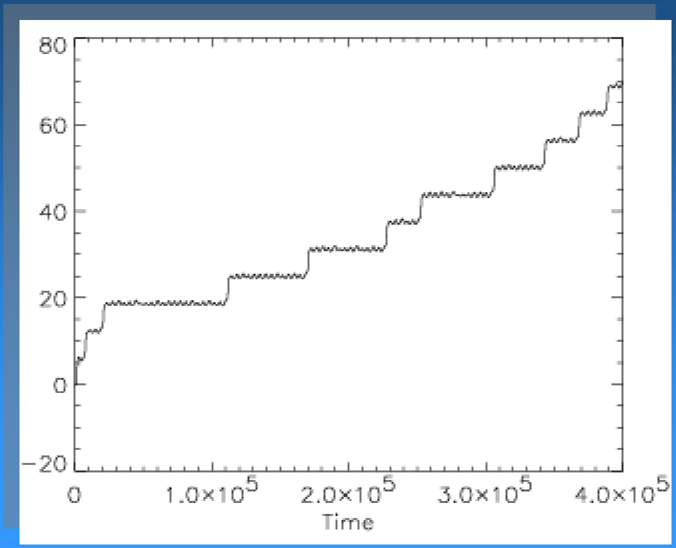
Discontinuity in the x variable: $x(t_0^+) = x(t_0^-) + p_0$

$p_0 < p_{th}$ Exponential Decay

$p_0 > p_{th}$ Pulse solution

Reinjection mechanism: 2-dimensional phase-space

Adler equation is a *bistable* system (in [0,2π])
or *multistable* in R^1 (jumps of 2π)



Fitzhugh-Nagumo system

Dynamical Regimes

$$\dot{x} = y + x - \frac{x^3}{3}$$

$$\dot{y} = -\varepsilon(x - a)$$

$a \epsilon \mathbb{R}$ and $\varepsilon \ll 1$

a : Control parameter

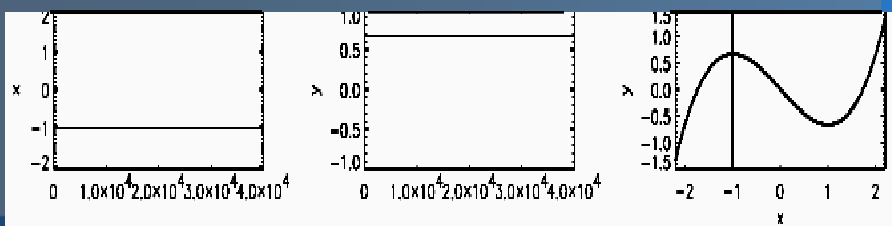
Single steady state : $(x_s , y_s) = (a , -a + a^3 /3)$

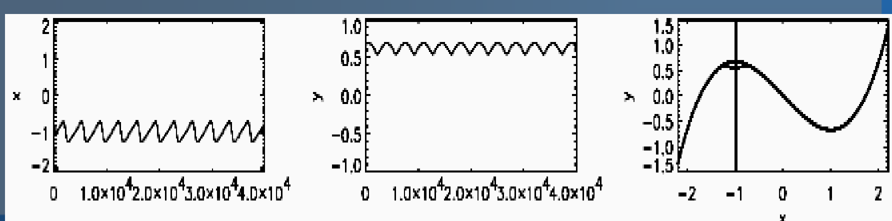
Stable for $| a | > 1$

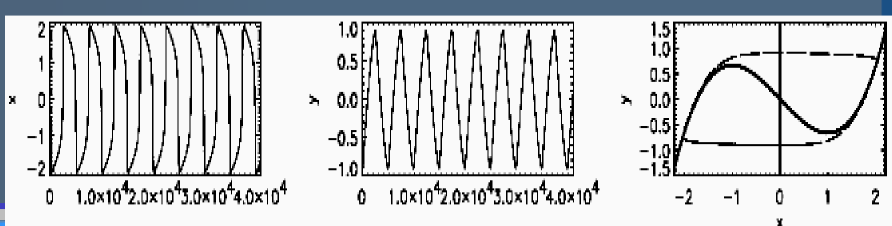
Unstable for $| a | < 1$

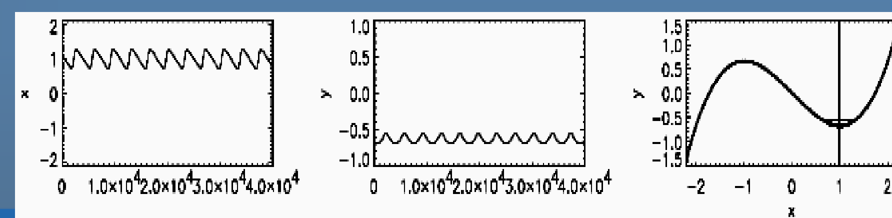
At the critical value $| a_c| = 1$ Supercritical Hopf Bifurcation

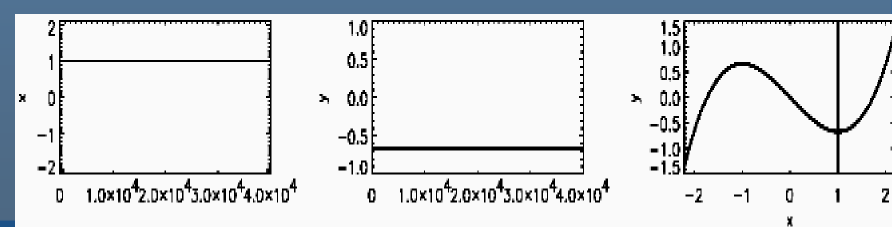
$\Rightarrow$ harmonic oscillations ($T = 2\pi / \varepsilon^{1/2}$)











Increase of the control parameter ***a***
from ***a** = -1.01* to ***a** = 1.01*

Fitzhugh-Nagumo system

Excitability

Important feature: Threshold-like response to external perturbations

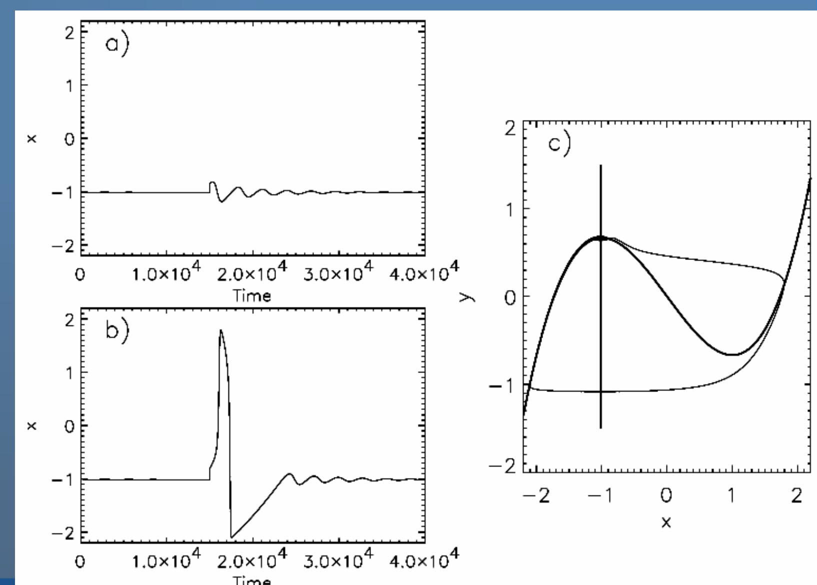
$$\dot{x} = y + x - \frac{x^3}{3}$$

$$\dot{y} = -\varepsilon(x - a)$$

$$+ p_0\delta(t - t_0)$$

$p_0 < p_{th} \Rightarrow$ Response proportional to the perturbation

$p_0 > p_{th} \Rightarrow$ Response independent of the perturbation



Singular Perturbation Theory

ε small ⇒ separation of the system evolution in two time scales: $O(1)$ and $O(\varepsilon)$

$$\dot{x} = y + x - \frac{x^3}{3}$$

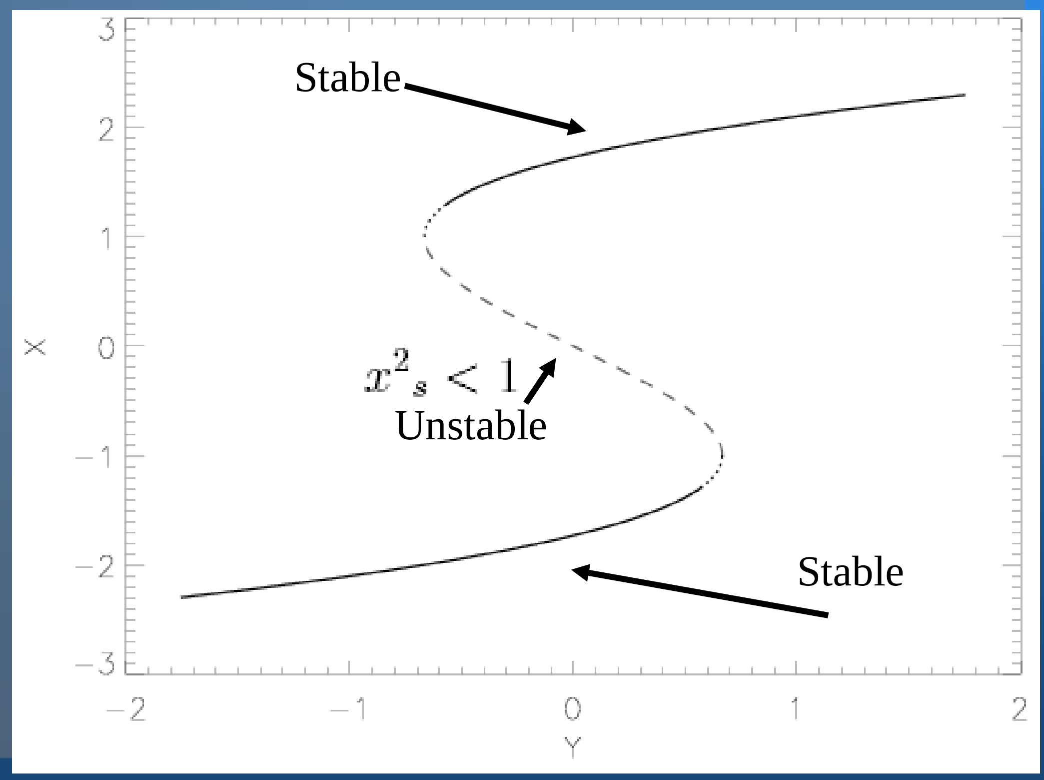
$$\cancel{\dot{y} = -\varepsilon(x - a)}$$

Fast Subsystem

y as a bifurcation parameter

Fixed Points $y + x_s - \frac{x^3{}_s}{3} = 0$

Linear stability analysis



Singular Perturbation Theory

$\boxed{\text{Slow Evolution}}$ $\Rightarrow$ x instantaneously follows y variations

By means of the time-scale change $\tau = \varepsilon\, t$ and putting $\varepsilon = 0$

$$0 = y + x - \frac{x^3}{3}$$
$$\dot{y} = -(x - a)$$

$\longleftarrow$ Fixed points of the fast subsystem define the branches of slow motion (Slow manifold)

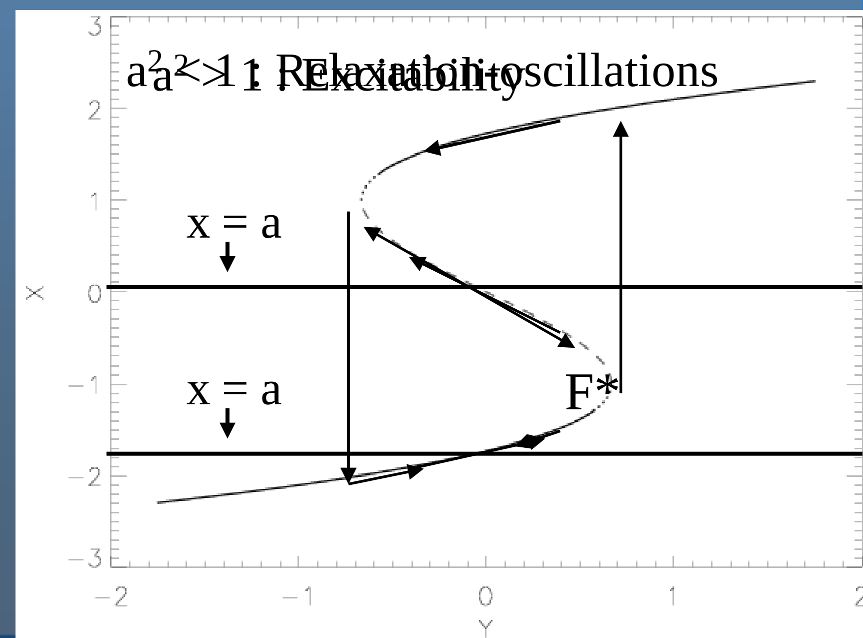
Stable points $\Rightarrow$ Attractive branches

Unstable points $\Rightarrow$ Repelling branches

If x $> a \Rightarrow d_t y < 0$

If x $< a \Rightarrow d_t y > 0$

This define the flux direction on the cubic



Point F acts as excitability threshold

Fitzhugh-Nagumo system

Noise-effects: Random Pulses

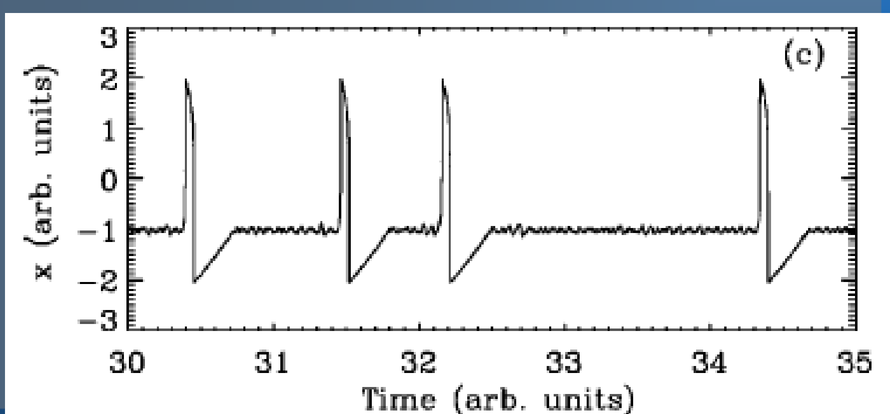
$$\dot{x} = y + x - \frac{x^3}{3}$$

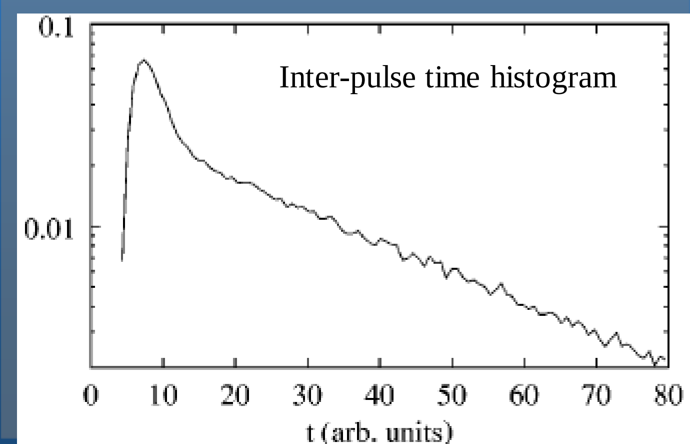
$$\dot{y} = -\varepsilon(x-a) + \xi(t)$$

$\xi(t)$ zero mean δ-correlated gaussian noise

If the system is in its stable fixed point:
Random events $\Rightarrow$ Kramers' law

Noise fluctuations erratically overcome the excitability threshold





Fitzhugh-Nagumo system

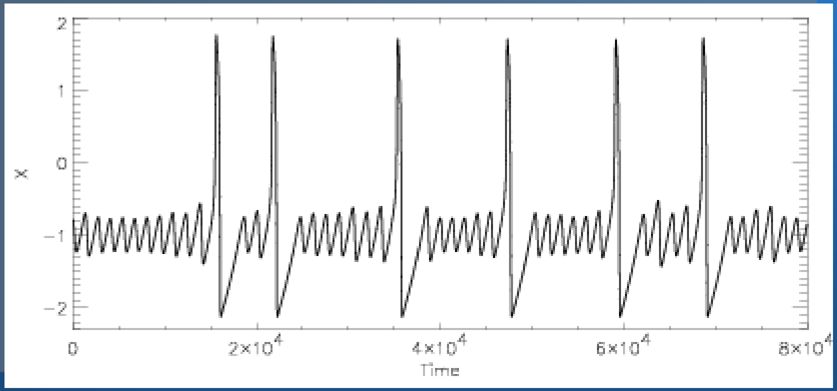
Noise-effects: Stochastic "phase-locking"

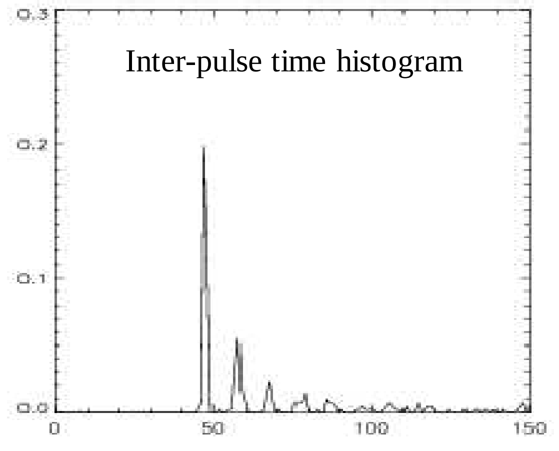
$$\dot{x} = y + x - \frac{x^3}{3}$$

$$\dot{y} = -\varepsilon(x - a) + \xi(t)$$

$\xi(t)$ zero mean δ-correlated gaussian noise

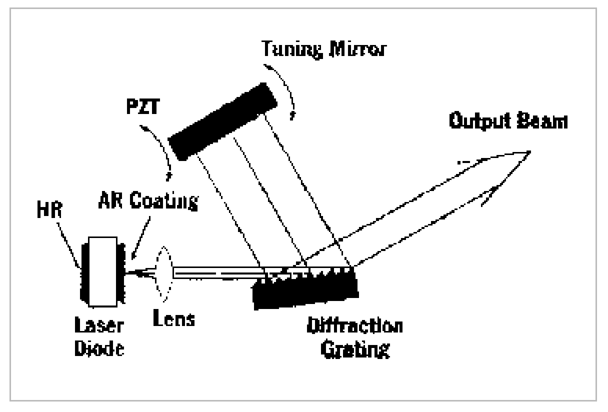
In the quasi-harmonic regime:
Multimodal probability distribution
(peaks at nT_{harm} , n =1,2...)





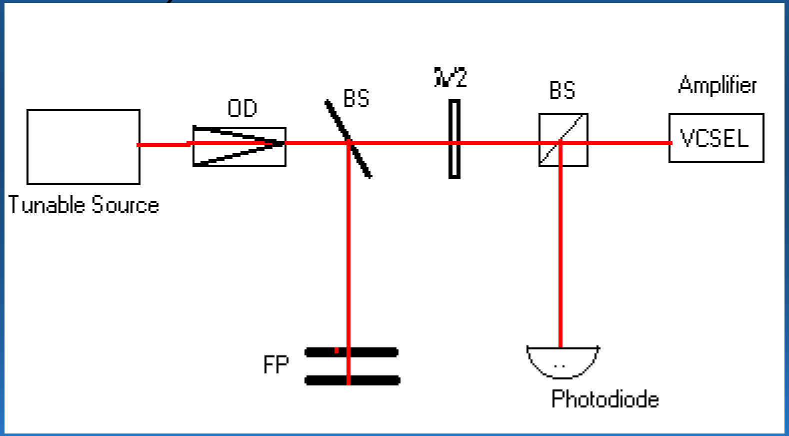
Modulation of the excitability threshold $\Rightarrow$ Modulation of emission probability

Fitzhugh-Nagumo dynamics in VCSOAs



Schematic drawing of the tunable source:

a λ/2 wave-plate allows us to optimize the feedback strenght.



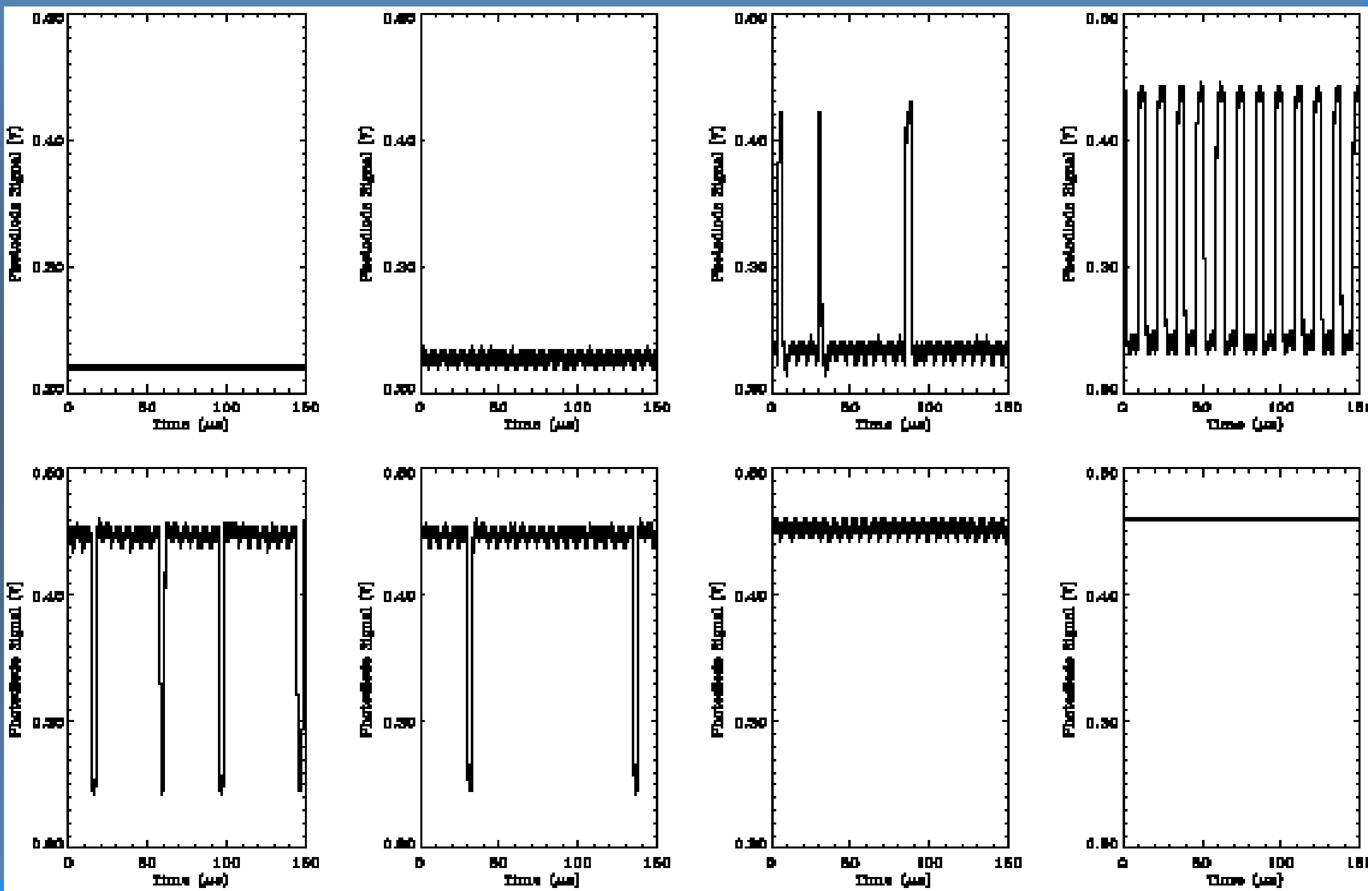
Optical injection:
The amplifier is a 10 μm diameter VCSEL, emitting at 980 nm, operating below threshold.

The control parameters are: The slave pump current, the injected wavelength and the injected power.

Fitzhugh-Nagumo dynamics in VCSOAs

Dynamical Regimes

Output power for a fixed injected frequency and power, and increasing pump current values



I =179.9 mA →180.6 mA
Power =0.8 mW

λ=982.3 nm (Fixed in the experiment)

Stable State

Limit Cicle

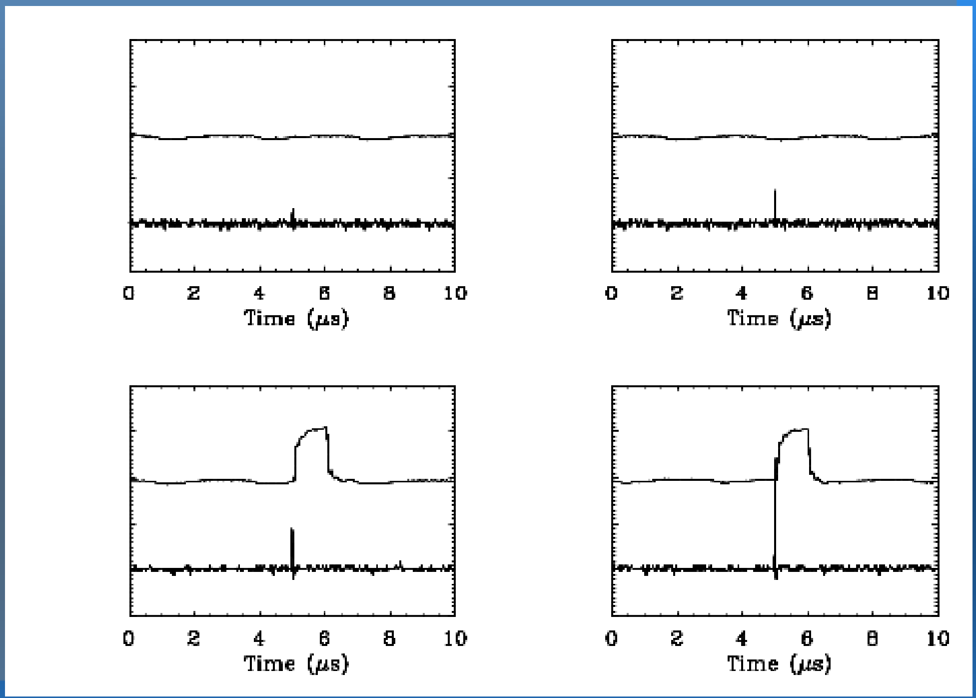
Random Pulses

Relaxation Oscillations

The sequence is similar to the transition between excitable and self-oscillatory regime in a noisy Fitzugh-Nagumo system as the parameters are varied.

Fitzhugh-Nagumo dynamics in VCSOAs

Excitability Test



We apply to the pump current a pulse (8 ns) of variable amplitude.
The control parameters are fixed.

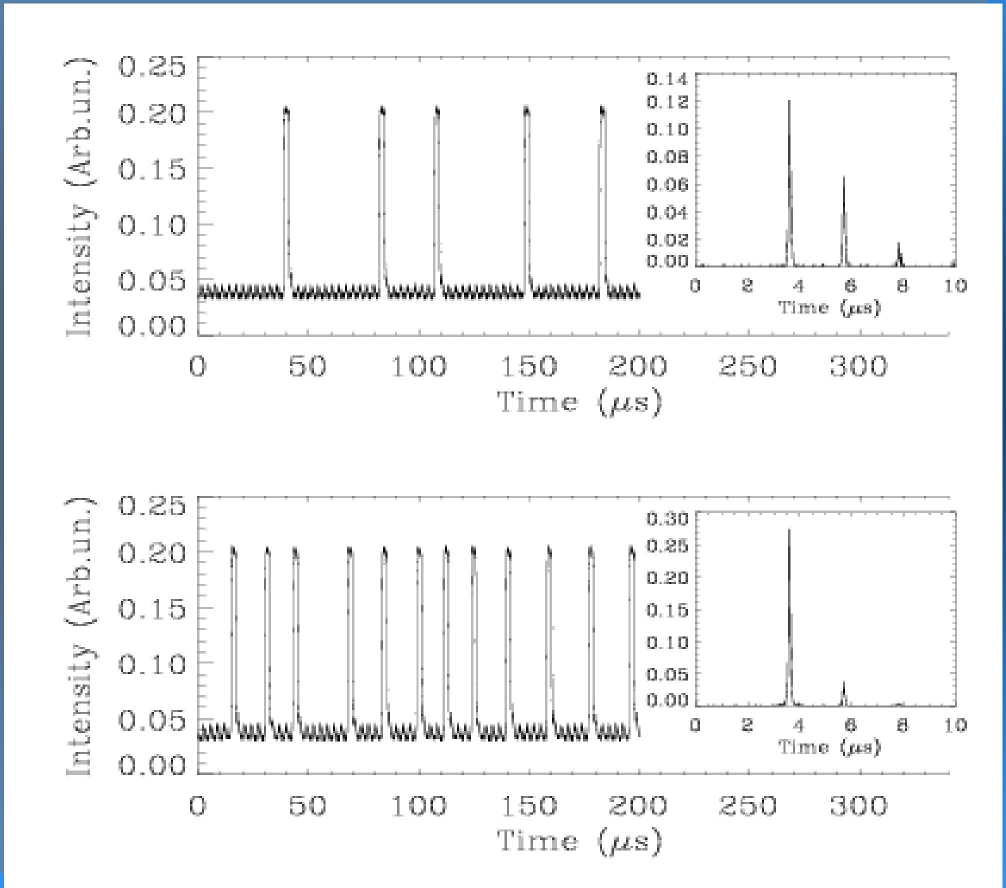
Perturbation Amplitudes:
100, 200, 300, 800 mV.

In an excitable system a perturbation larger than a certain threshold triggers the emission of a pulse whose amplitude and duration are independent on the perturbation itself.

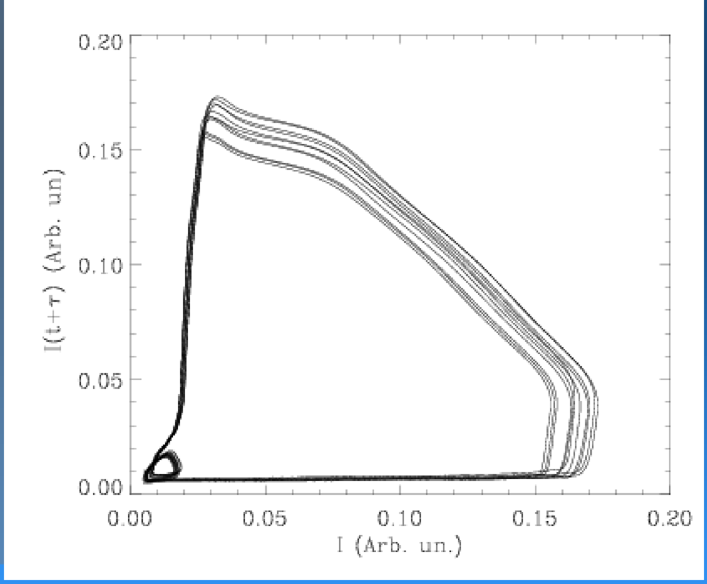
Fitzhugh-Nagumo dynamics in VCSOAs

Noise effects above the Hopf bifurcation

In a certain region of the parameter space, the system shows a limit cicle: in these conditions we add noise to the pump current via a bias-T.



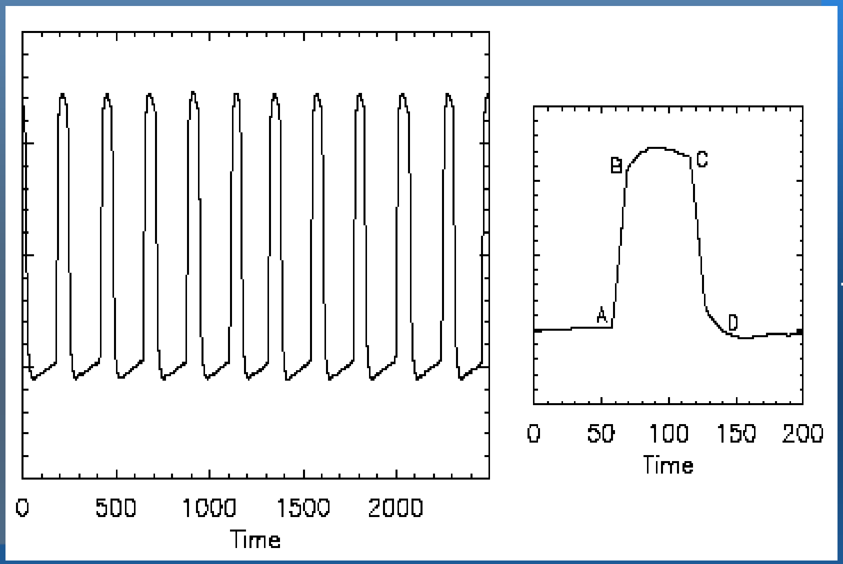
Incresing noise



The excitable pulses are “synchronized” to the quasi-harmonic limit cicle frequency.

Fitzhugh-Nagumo dynamics in VCSOAs

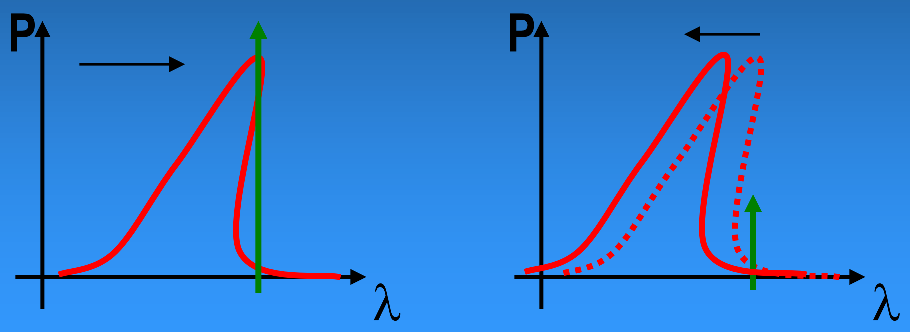
Physical mechanism



Branch AB: Because of the incresed intracavity field, the gain (red)-shifts and the output intensity abruptly increases.

Branch BC: The same interaction makes the temperature vary (increase of dissipated power) ⇒ refractive index change ⇒ cavity resonance shift.

Branch CD: Due to the drift of parameters, the output power drops and the process begins one more time.



Fitzhugh-Nagumo dynamics in VCSOAs

Physical Model

$$\partial_t E_\pm \pm v\partial_z E_\pm = \frac{v}{2}\left[i\Gamma\frac{\Omega}{cn}\chi(\Omega,N) - \alpha_{int}\right] E_\pm ,$$
$$\partial_t N = \frac{I}{eV_a} - R_{sp}(N) + v\frac{\Omega}{cn} Im\left[\chi(\Omega,N)\right]\left(|E_+|^2 + |E_+|^2\right) ,$$

$$\begin{aligned} E_+(0,t) &= r_1 E_-(0,t) + t_1' E_i , \\ E_-(L,t) &= r_2 e^{i2\beta L} E_+(L,t) , \\ E_r &= r_1' E_i + t_1 E_-(0,t) , \\ E_t &= t_2 e^{i\beta L} E_+(L,t) , \end{aligned}$$

$$\beta \equiv n\Omega/c$$

n≡n(T) and the temperature of the device depends on the dissipated power:

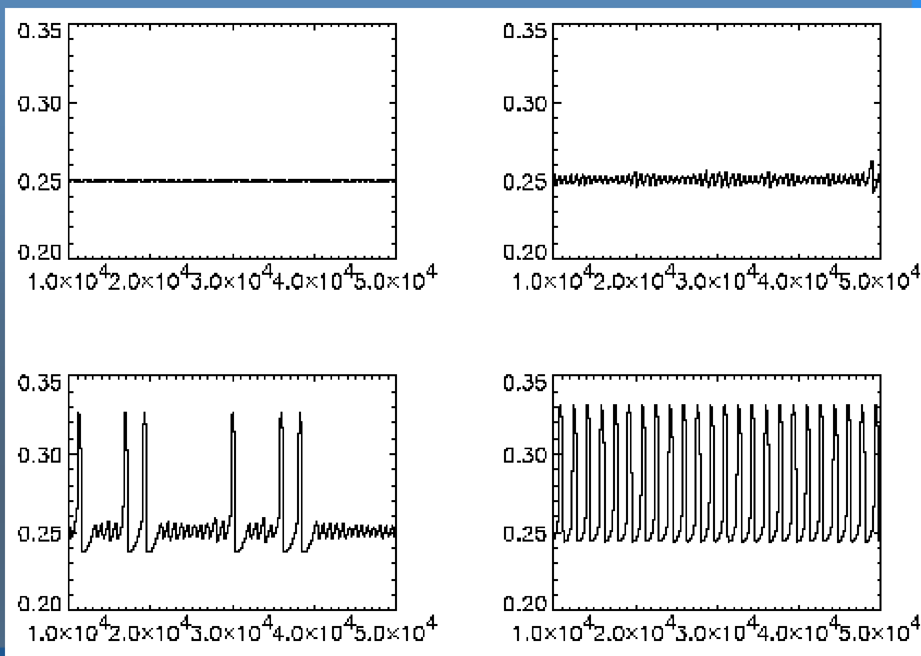
$$P = VI + A\left(|E_i|^2 - |E_r|^2 - |E_t|^2\right) - B < R_{sp}(N) > \quad \Rightarrow \quad \partial_t T = -\gamma_{th}\left(T - T_s - R_{th}P\right) ,$$

- Neglect internal losses: $\alpha_{int} = 0$
- Linearize the spontaneous emission rate: $R_{sp}(N) = \gamma_e N$
- Simplest description of the susceptibility: $\chi = - a\,(N - N_t)(\alpha + i)$
- Linearize the temperature dependence of the effective index: $n(T) = n_0 + b(T - T_0)$
- Adiabatic elimination of the forward and backward fields

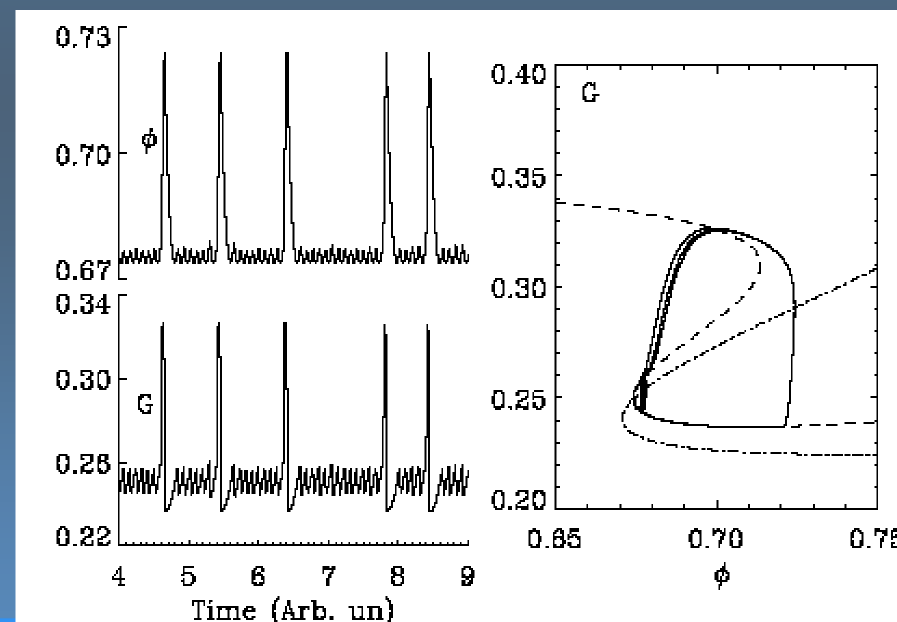
$$\begin{aligned} \partial_t G &= \gamma_{sp}(G_0 - G - P_i Q) , \\ \partial_t \phi &= -\gamma_{th}(\phi - \phi_e + \lambda G + \mu P_i Q) , \\ Q &= \frac{(1 - R_1)(1 + R_2 e^G)(e^G - 1)}{1 + R_1 R_2 e^{2G} - 2\sqrt{R_1 R_2}\, e^G cos(\phi - \alpha G)} \end{aligned}$$

Fitzhugh-Nagumo dynamics in VCSOAs

Physical Model: dynamical regimes



Time traces of the variables and corresponding orbit in the phase-space



We change the injected power.
The other control parameters are fixed.

The same sequence can be obtained by changing the pump current.

High-dimensional excitable systems

Fitzhugh-Nagumo with inertial term

$$\dot{x} = v$$
$$k\dot{v} + v = y + x - \frac{x^3}{3}$$
$$\dot{y} = -\varepsilon(x - a)$$

$k\epsilon\ (1,5]\ ;\ \varepsilon \ll 1;\ a\ \epsilon\ \mathbb{R}$

Single steady state : (xs , ys) = (a , -a + a3 /3)

Linear Stability analysis

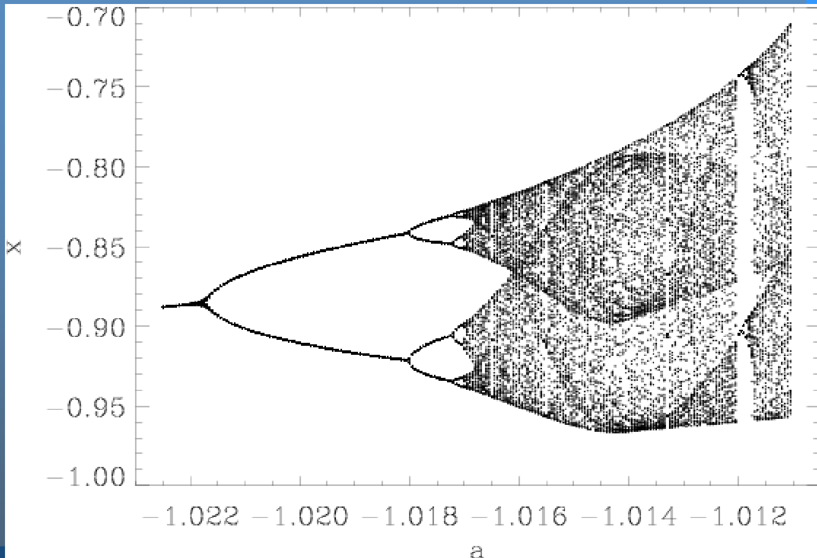
$$\dot{\delta x} = \delta v$$
$$k\dot{\delta v} = -\delta v + \delta y + (1 - a^2)\delta x$$
$$\dot{\delta y} = -\varepsilon\delta x$$

$$k\lambda^3 + \lambda^2 + (a^2 - 1)\lambda + \varepsilon = 0$$

$$\omega = \pm\sqrt{\varepsilon}$$
$$a = \pm\sqrt{1 + \varepsilon k}$$

Dynamical behaviour far from the bifurcation?

Dynamical Regimes: Period-doubling



The usual Hopf bifurcation is now followed by a period-doubling route to a small-amplitude chaotic attractor

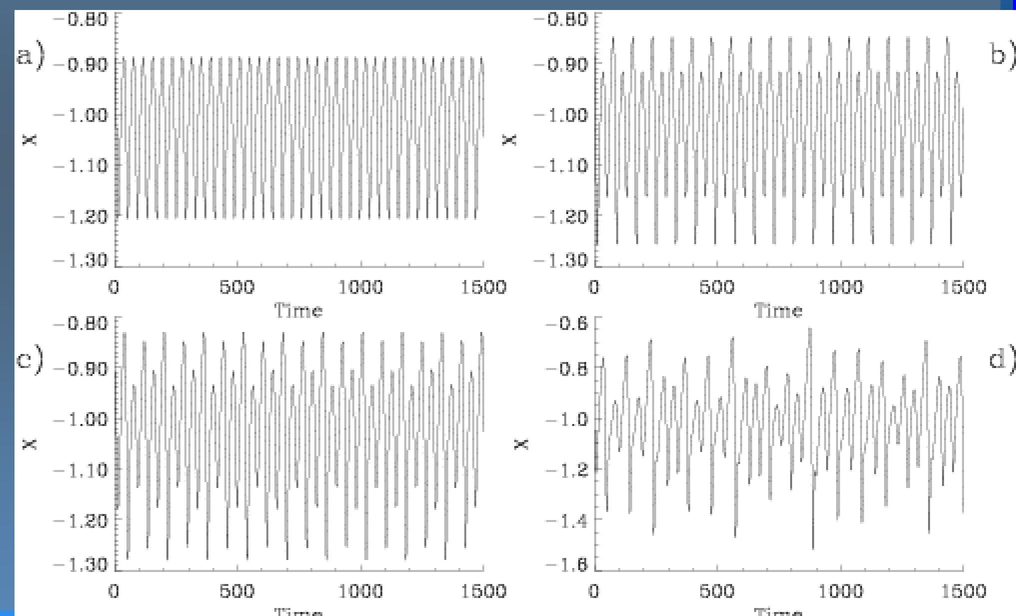
Temporal evolution of the variable x increasing the control parameter *a*:

a = -1.0225 Period 1

a = -1.0190 Period 2

a = -1.0172 Period 4

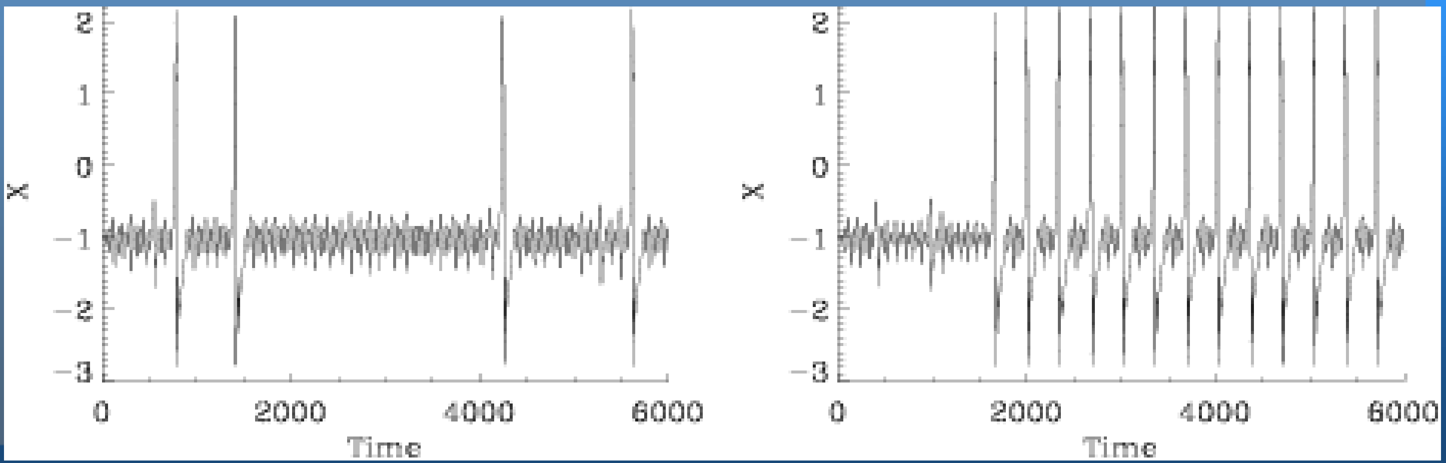
a = -1.0100 Chaos



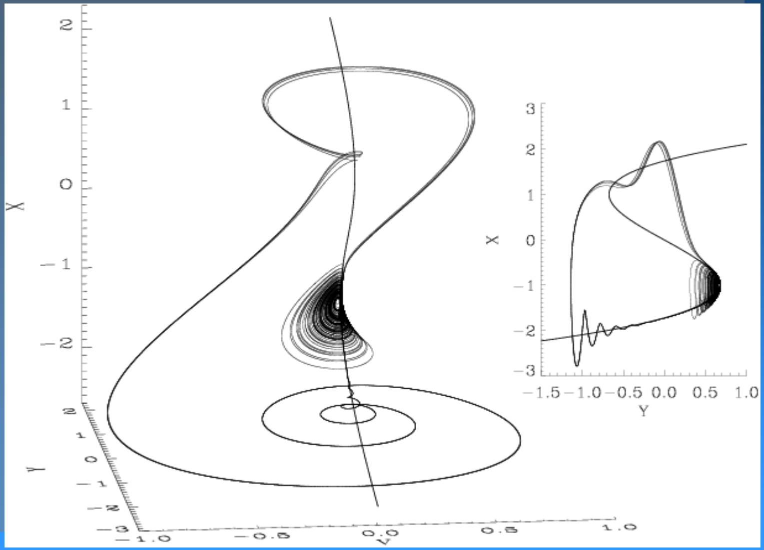
Further increasing *a* the mean amplitude of the chaotic attractor increases

Chaotically Spiking Canards and Oscillations

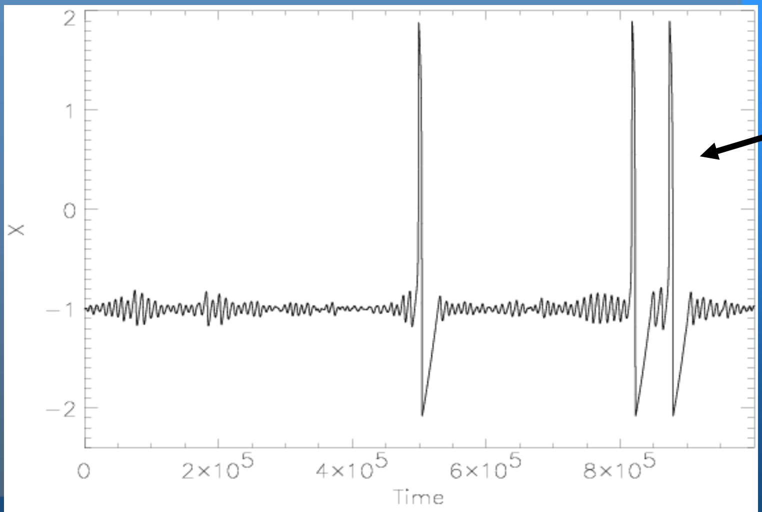
Random sequence of pulses on a chaotic basal state



1. Fixed *a*, spikes sequences corresponding to different initial conditions are unrelated.
2. Further increasing *a* the rate of random spikes increases until that a regime of periodic oscillations is found



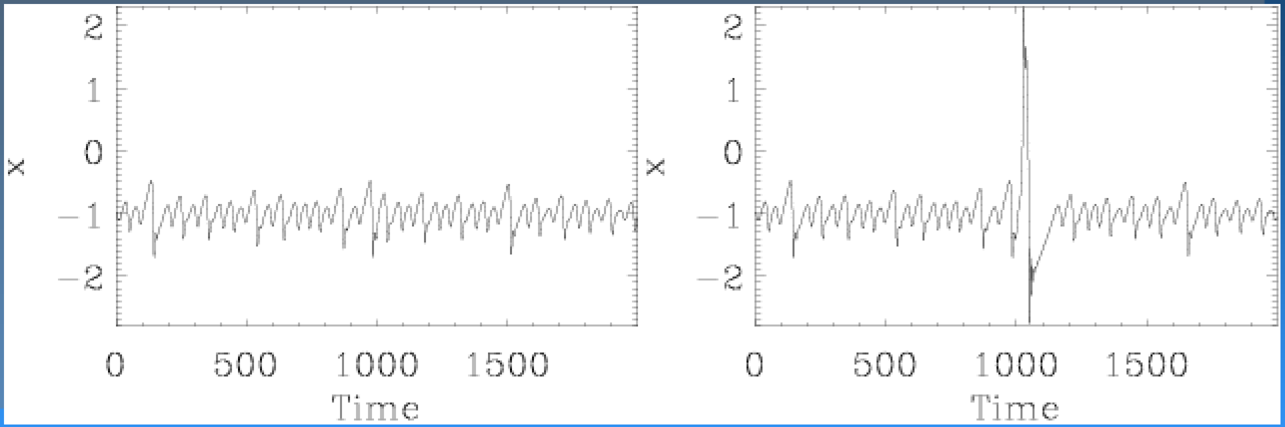
Chaotic Spiking and Excitability



Similar to FN with additive noise:
the noisy background triggers excitable pulses

The deterministic aperiodic background can trigger excitable pulses if overcomes a certain threshold

Excitability Test: The system is prepared in its chaotic attractor (without spikes)



For a sufficiently strong stimulus the system recovers its initial attractor following a deterministic orbit that does not depend on the details of the perturbation

Singular Perturbation Theory

ε small ⇒ separation of the system evolution in two time scales: $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$

Fast Subsystem

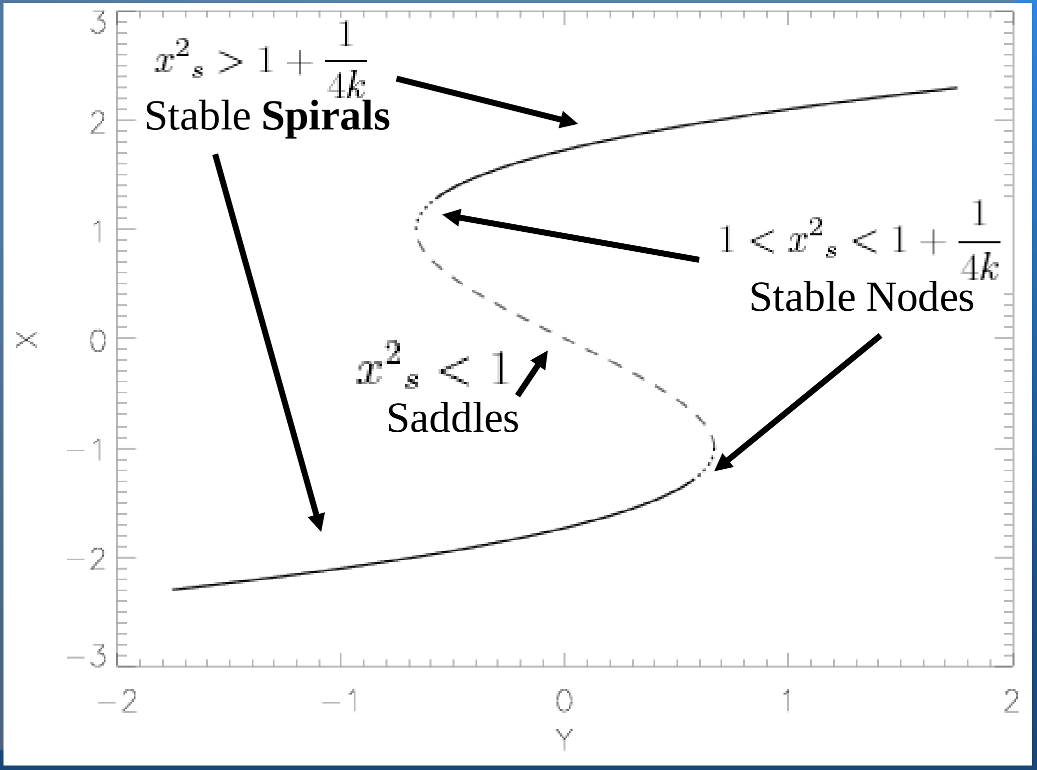
$$\dot{x} = v$$
$$k\dot{v} + v = y + x - \frac{x^3}{3}$$
$$\dot{y} = 0$$

⇒ y as a bifurcation parameter

Fixed Points

$$y + x_s - \frac{x^3{}_s}{3} = 0$$

$$\lambda_{1,2} = \frac{1}{2k}\left(-1 \pm \sqrt{1 - 4k(x^2{}_s - 1)}\right)$$



Singular Perturbation Theory

Slow Evolution $\Rightarrow$ x instantaneously follows y variations

By means of the time-scale change $\tau = \varepsilon\, t$ and putting $\varepsilon = 0$

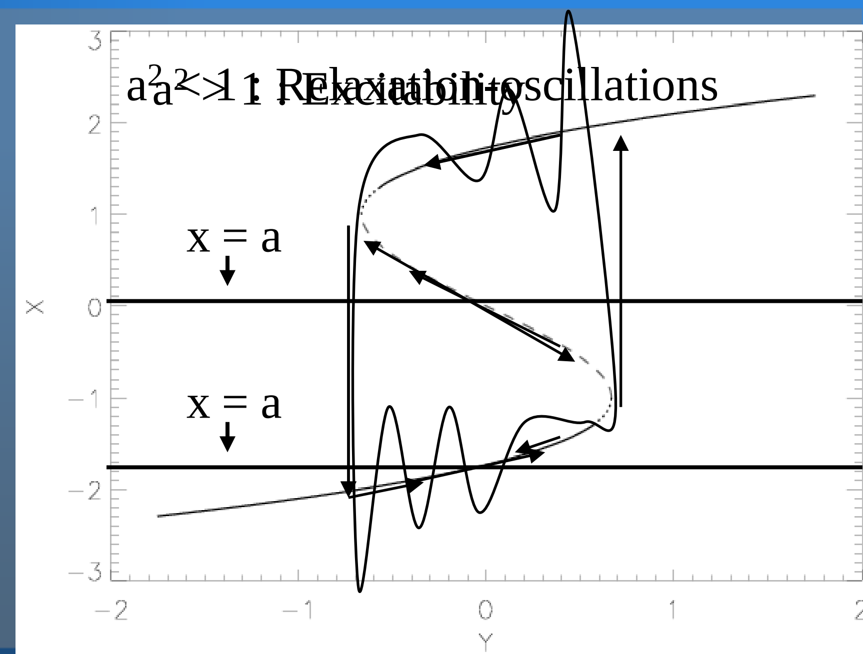
$$0 = y + x - \frac{x^3}{3}$$

$$\dot{y} = -(x - a)$$

Defines the branches of slow motion (Slow manifold)

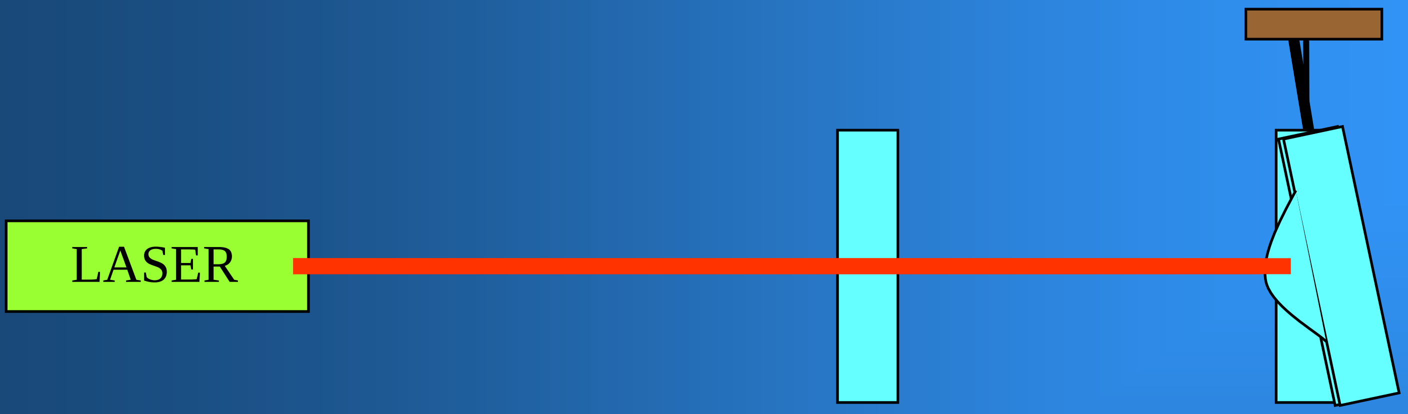
If $x > a \Rightarrow d_t y < 0$
If $x < a \Rightarrow d_t y > 0$

Whenever jumps from one stable branch to the other occurs, the system is “spiralling” around the attractive part of the slow manifold



Physical Example: Optomechanical Resonator

Optical injection on the long-wavelength side of the cavity resonance

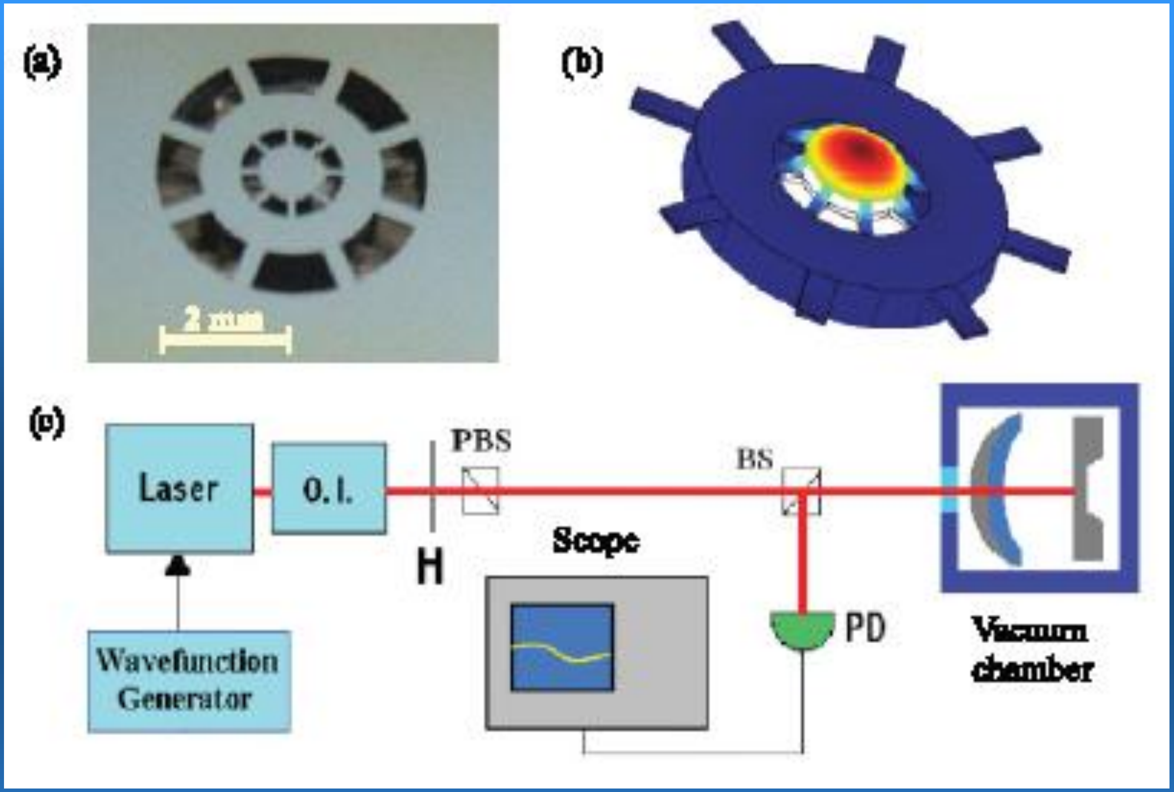


Radiation-pressure driven cavity: The radiation pressure tends to increase the cavity lenght respect to the cold cavity value⇒ the intracavity optical power increases

Photo-thermal expansion: The increased intracavity power *slowly* varies the temperature of mirrors ⇒ heating induces a decrease of the cavity length through thermal expansion

1. Competition of the effects leads to Fitzhugh-Nagumo dynamics
2. The mechanical reaction of the pendulum mirror is responsible for the inertial features of the fast motion

Optomechanical Resonator: Experimental Setup



Optomechanical resonator: Physical Model

We write the cavity lenght variations as $L(t) = L_{rp}(t) + L_{th}(t)$

Radiation Pressure Effect

Limit of small displacements$\Rightarrow$
Damped oscillator forced by the intracavity optical power

$$\ddot{L}_{rp} + \frac{\Omega}{Q}\dot{L}_{rp} + \Omega^2 L_{rp} = \frac{2P_c}{mc}$$

$$P_c = \frac{A P_{in}}{1 + \left\{\frac{4F}{\lambda}[L(t) + L_0]\right\}^2}$$

Photothermal Effect

The temperature relaxes towards equilibrium at a rate ε and $L_{th} \propto T$

$$\dot{T} = -\varepsilon\left(T - T_0 - \frac{dT}{dP_c}P_c\right)$$

$$\dot{L}_{th} = -\varepsilon\left(L_{th} + \left|\frac{dL_{th}}{dP_c}\right|P_c\right)$$

Physical model

$$\dot{\phi} = v\,,$$

$$\dot{v} + \frac{1}{Q}v = -\phi + \frac{\alpha}{1+(\delta_0+\phi+\theta)^2}$$

$$\dot{\theta} = -\varepsilon\left[\theta + \frac{\beta}{1+(\delta_0+\phi+\theta)^2}\right].$$

$$\delta_0 = L_0/\Delta,\ \alpha = \tilde{\alpha} A P_{in},\ \beta = \tilde{\beta} A P_{in}$$

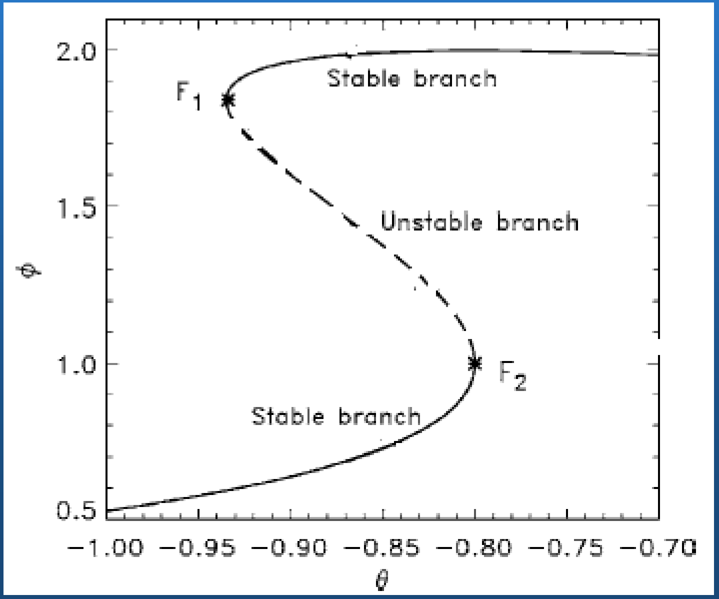
Stationary solutions

$$v_s = 0$$

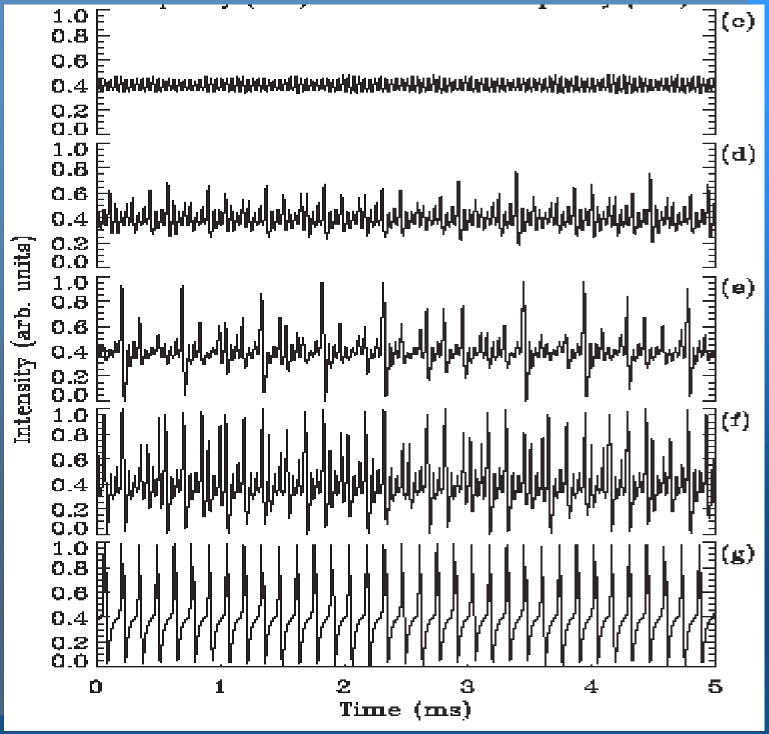
$$-\phi_s[1+(\delta_0+\phi_s+\theta_s)^2]+\alpha = 0$$

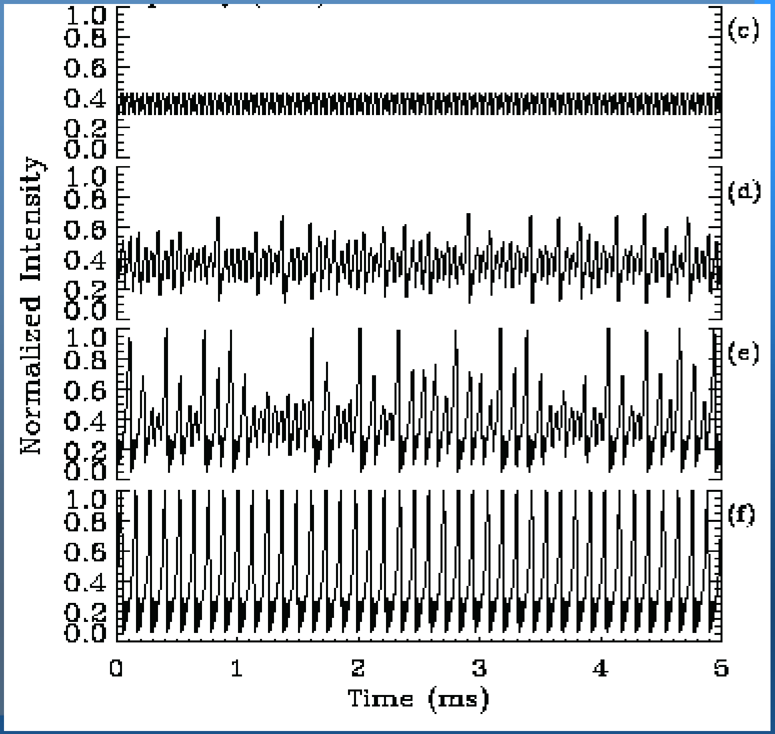
$$\theta_s = -(\beta/\alpha)\phi_s$$

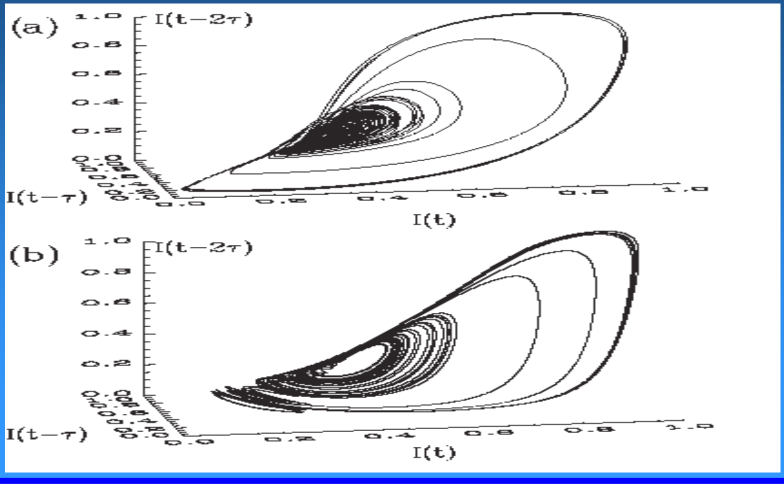
Cubic-like manifold
Slow and fast time scales



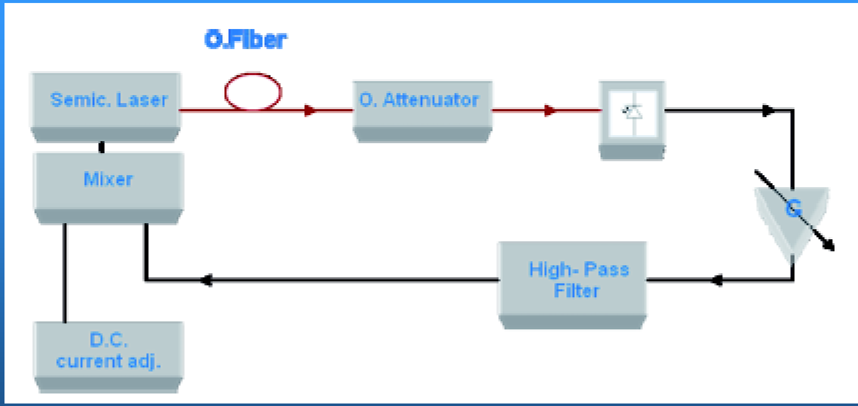
Experimental vs Numerical Results







Semiconductor lasers with optoelectronic feedback



$$\dot{S} = [g(N - N_t) - \gamma_0]\, S$$

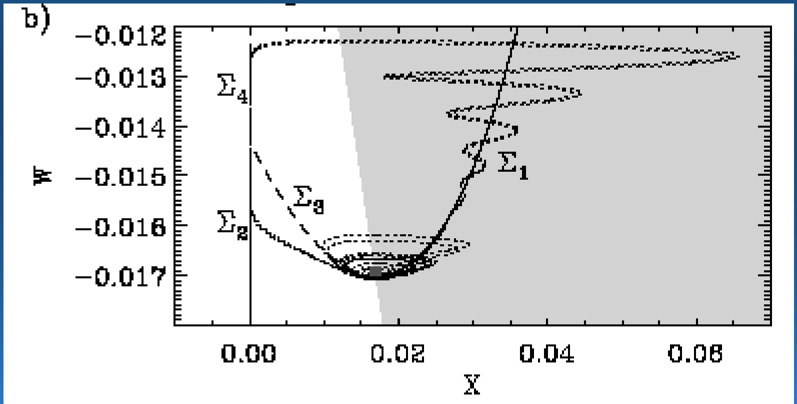
$$\dot{N} = \frac{I_0 + f_F(I)}{eV} - \gamma_c N - g(N - N_t)\, S$$

$$\dot{I} = -\gamma_f I + k\dot{S}$$

$$\dot{x} = x(y-1),$$

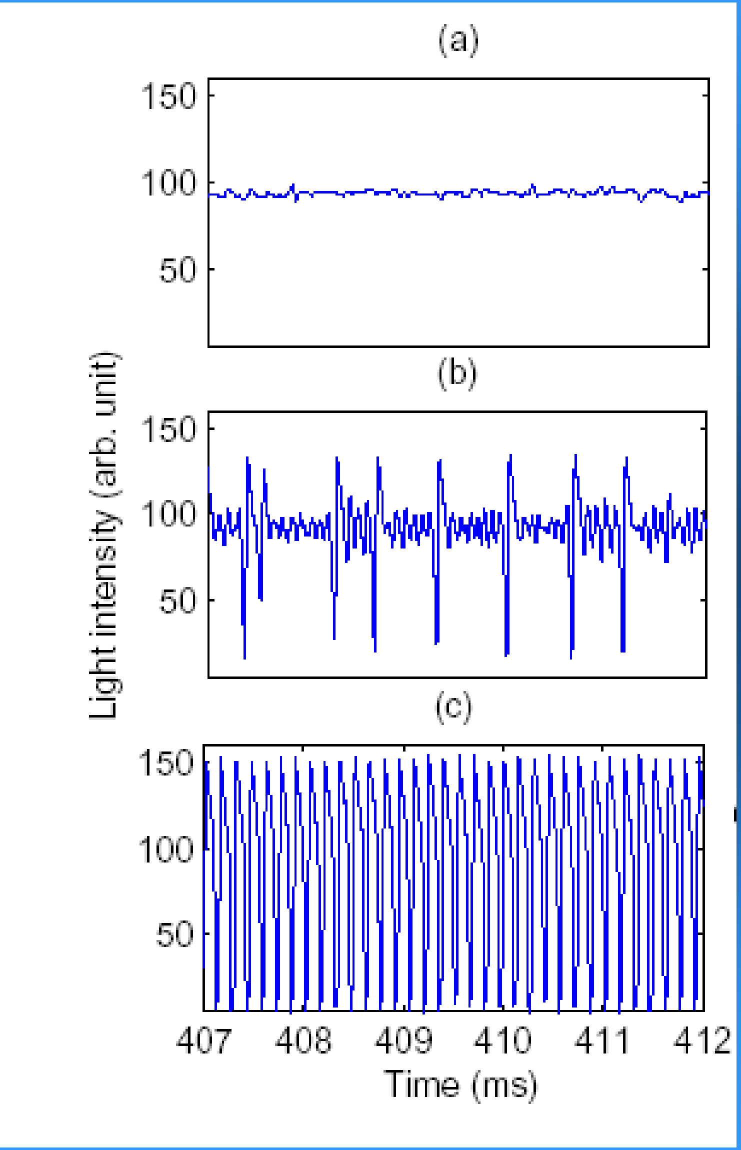
$$\dot{y} = \gamma(\delta_0 - y + f(w+x) - xy)$$

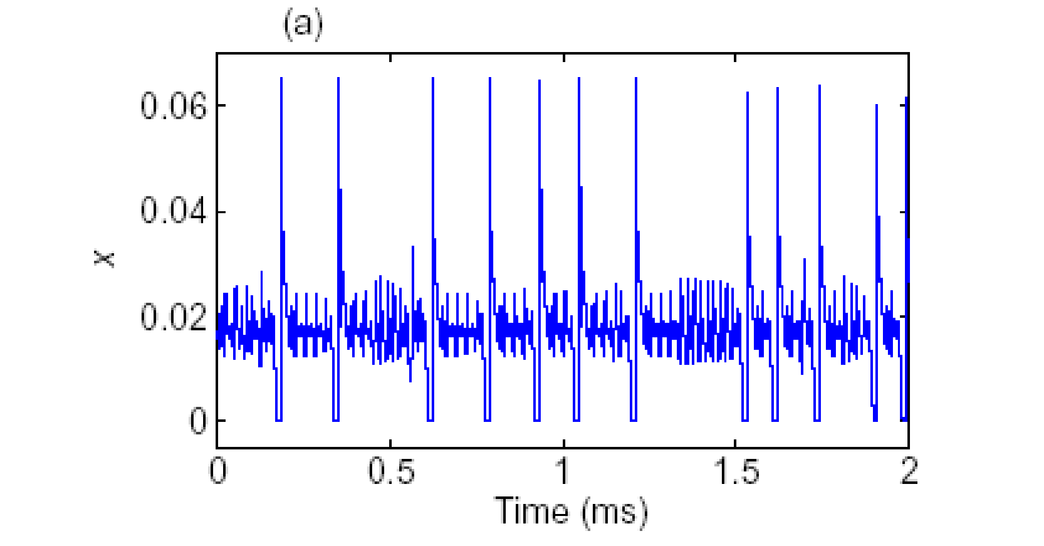
$$\dot{w} = -\epsilon(w+x),$$



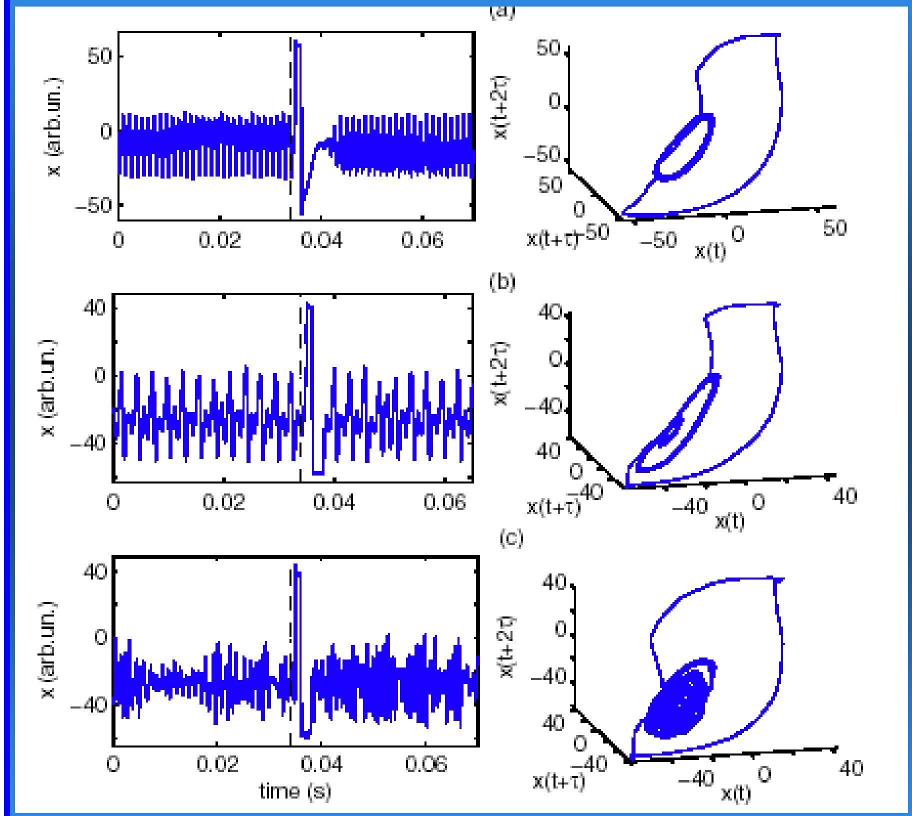
Cubic-like manifold
Slow and fast time scales

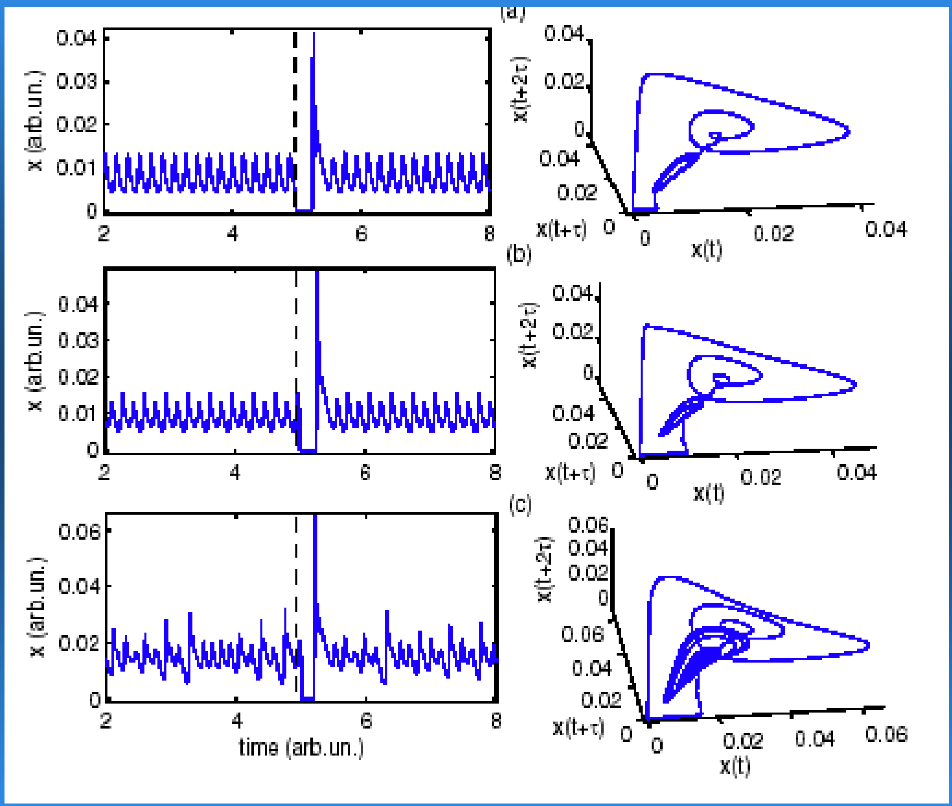
Semiconductor lasers with optoelectronic feedback





Semiconductor lasers with optoelectronic feedback





Spatially extended excitable media

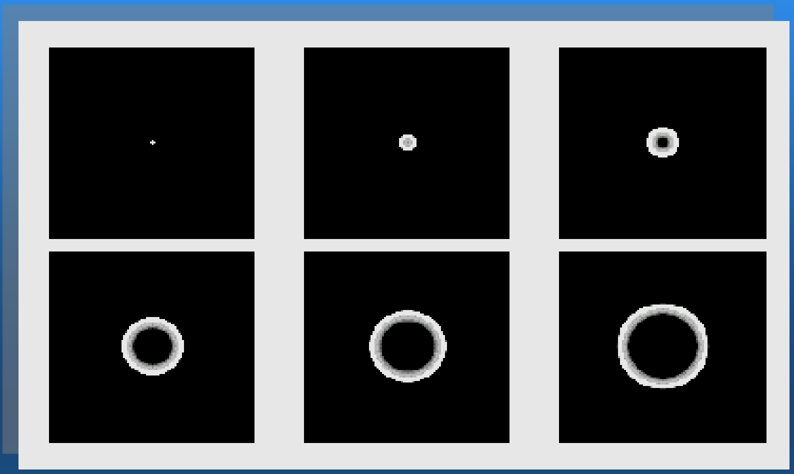
Spatially extended Fitzhugh-nagumo

Spatially extended excitable systems

Local excitable dynamics + diffusion term

$$\partial_t u = v + u - \frac{u^3}{3} + D\nabla^2 u\,,$$
$$\partial_t v = -\varepsilon(u - a)$$

$\boxed{+\, p_0\delta(x - x_0,\ t - t_0)}$



❖ Excitable waves are dynamical states arising from the propagation through the whole system of a locally induced nonlinear response

Small size optical amplifiers (transv. dim. < 10 μm) → Excitable systems

Broad area optical amplifiers (100 μm) → Spatially extended media ?

Carriers and temperature diffusion

Excitable optical waves

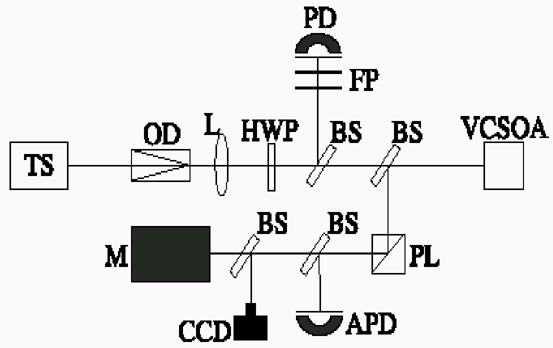
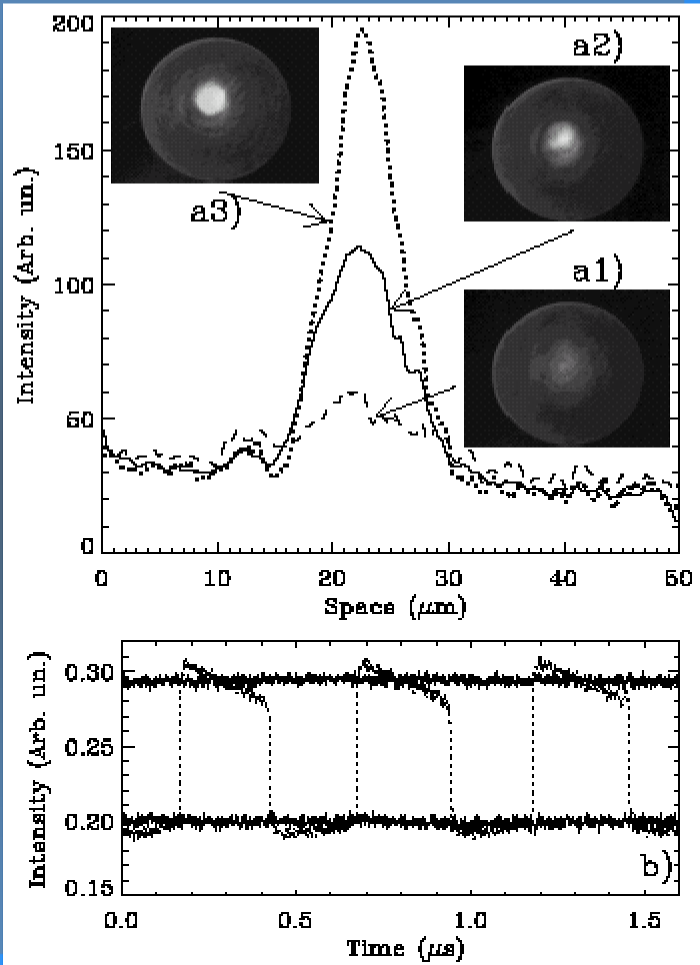


Figure 2.2: The experimental setup. TS: Tunable source, OD: Optical diode, HWP: Half-wave plate, L: Lens, BS: Beam-splitter, FP: Fabry-Perot interferometer, PD: Photodiode, PL: Linear polarizer, CCD: CCD camera, M: Monocromator.

Near-field images, transverse profiles and corresponding temporal series

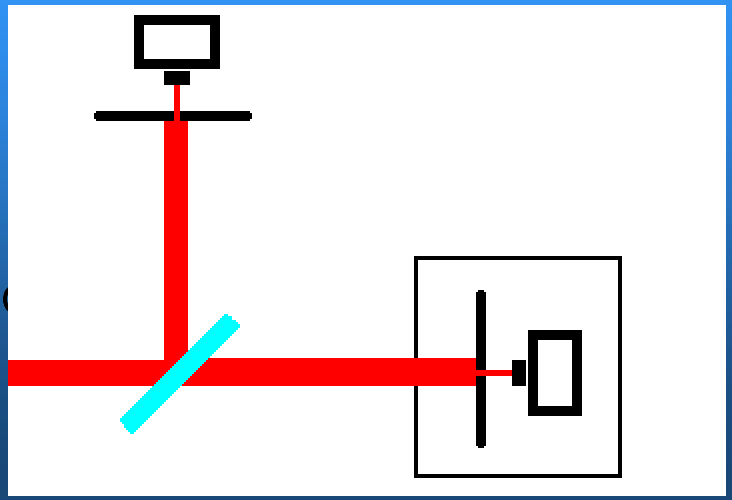


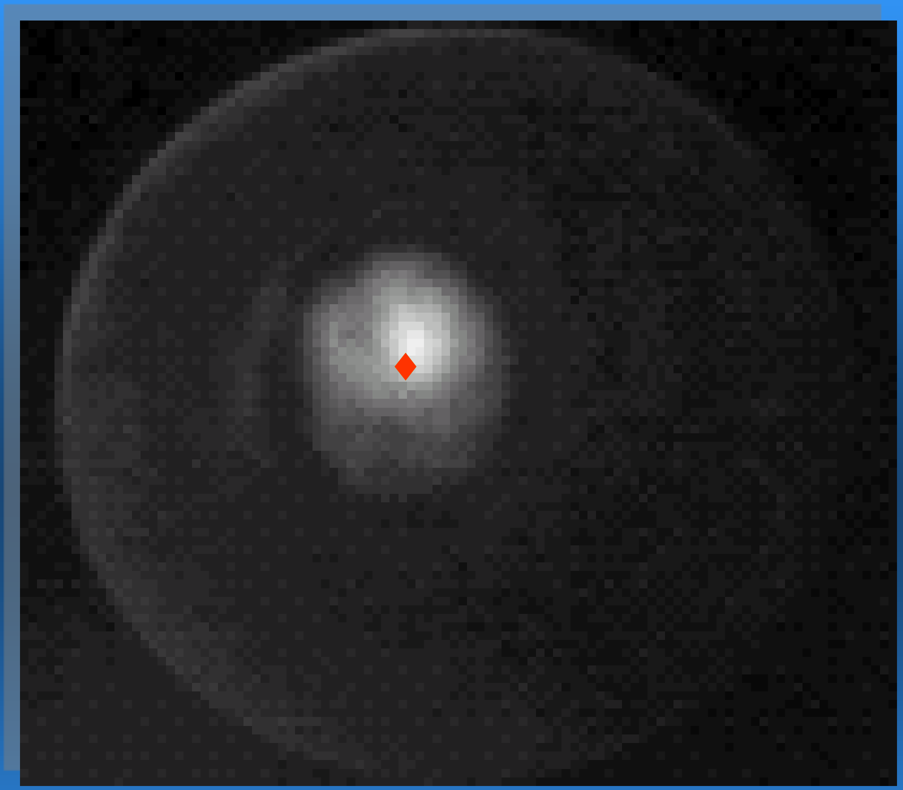
By increasing the VCSOA bias current we pass from a low stable state to a high stable state crossing a regime of self-sustained oscillations

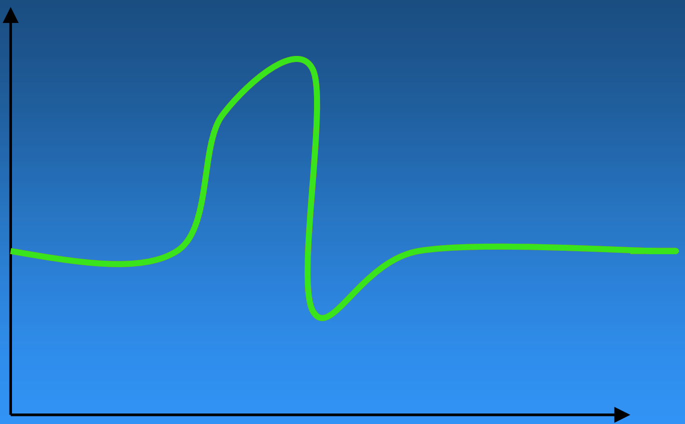
CCD cannot resolve dynamics faster than 1 ms

Excitable optical waves

Wave propagation: Detection System





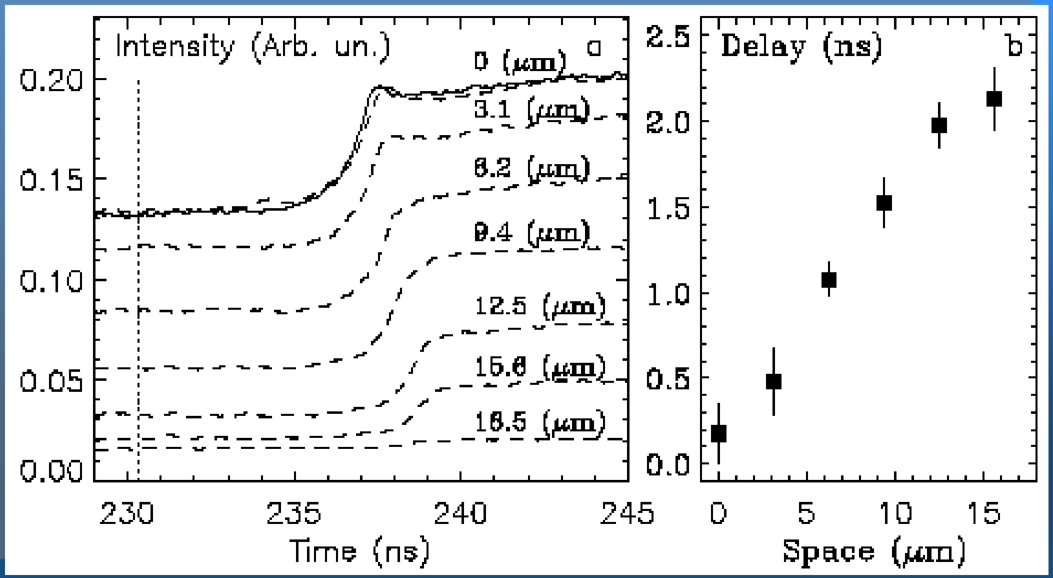


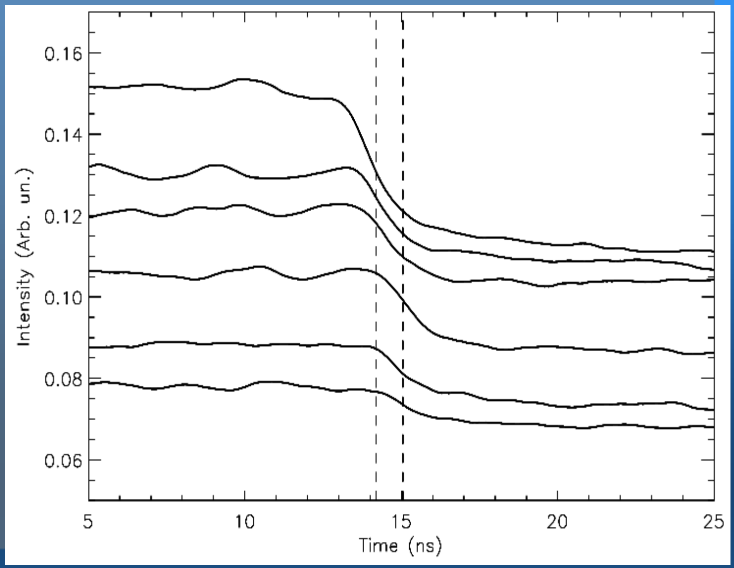
APD #1 (Fixed)
APD #2

If an excitable pulse is born at the center of the spot and then it propagates
we should observe a temporal delay between the signals

Excitable optical waves

Propagation velocity





Pulse duration > delays ⇒ We can not record the entire pulse without losing temporal resolution

- ✓ The delay is proportional to the distance between monitored points in the middle region then it saturates
- ✓ The amplitude of the pulses decreases
- ✓ The delays at the falling edges of the pulses are smaller ⇒ change in the refractory time during the propagation (Finite size effects)

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