Ministry of Higher Education and Scientific Research

Diyala University \ College of Science

Mathematic Department

Complex Analysis

Fourth Year

By

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403MACA Complex Analysis units 8

Theoretical 4hr/week

Tutorial 2hr/week

Practical - hr/week

1- Complex Numbers

complex number definition, properties Geometric representation, Root of complex number, field of complex number as metric field .

(12hrs)

2-Regions in The Complex Plane

Open set, Close set in a complex plan, connectedness, Region, smooth Carve.

(4hrs)

3-Analytic Function

Function of a complex Variable, Limits, Continuity, Derivatives, Cauchy- Riemann Equations, Analytic Function, Harmonic Functions.

(16hrs)

4- Elementary Functions

Exponential Function, Trigonometric Function, Logarithmic Function, Hyperbolic Functions.

(8hrs)

5-Serise

Convergence of Sequence, Convergence of Series, Pour Series, Convergence Pour Series, Taylor Series, Laure Series.

(12 hrs)

6-Integrals

Definition Integrals of Function, Contour Integrals , Cauchy_Goursat Theorem, Cauchy_Integral Formula, Liouville's Theorem and the Fundamental Theorem of Algebra.

(16hrs)

7-Residues and Poles

Residues, Cauchy's Residue Theorem, Using a Single Residue, Singular Points, Zeros of Analytic Functions.

(12hrs)

8-Applications of Residues

Evaluation of Improper Integrals, Jordan's Lemma, Definit Integrals involving Sines and Cosines, Argument Principle , Rouch's Theorem.

(12hrs)

Complex analysis

Complex number:-

A complex number Z can be defined a ordered pairs $Z=(X,y)$ of real numbers x a y or $Z=X+iy$ where i is asymbol for $\sqrt{-1}$ we say that $X=Re(Z)$ and $y=im(Z)$ the imagin part

Fundmental opperaters at Complex numbers

① addition:

Let $Z_1=a_1+ib_1$, $Z_2=a_2+ib_2$ then

$$Z_1+Z_2=(a_1+ib_1)+(a_2+ib_2)=(a_1+b_1)+i(b_1+b$$

② subtraction:

$$Z_1-Z_2=(a_1+ib_1)-(a_2+ib_2)=(a_1-a_2)+i(b_1-$$

③ multiplication

$$Z_1 . Z_2=(a_1+ib_1).(a_2+ib_2)=(a_1a_2-b_1b_2) +i(a_1b_2+a_2b_1)$$

④ division

$$\frac{Z_1}{Z_2}=\frac{a_1+ib_1}{a_2+ib_2}=\frac{a_1+ib_1}{a_2+ib_2}.\frac{a_2-ib_2}{a_2-ib_2}$$

$$=\frac{a_1a_2+b_1b_2+i(a_2b_1-a_1b_2)}{a_2^2+b_2^2}$$

$$=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+i\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}$$

<u>Def</u> The Complex Conjugate of $Z = a + ib$ denoted

$\bar{Z} = \overline{a+ib} = a - ib$

and since $Z = a + ib$, $\bar{Z} = a - ib$ then $Z \cdot \bar{Z} = a^2 + b^2$

<u>The properties of $\bar{Z}$</u>

1- $\overline{Z_1 + Z_2} = \bar{Z}_1 + \bar{Z}_2$

2- $\overline{Z_1 \cdot Z_2} = \bar{Z}_1 \cdot \bar{Z}_2$

3. $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\bar{Z}_1}{\bar{Z}_2}$

4. $Z \cdot \bar{Z} = |Z|^2$

also $|Z^2| = |(x, y)^2| = |x + iy|^2$

$= \sqrt{x^2 + y^2}^2 = x^2 + y^2$

5. $\frac{Z + \bar{Z}}{2} = Re(Z)$

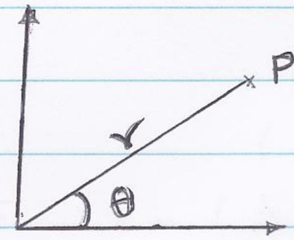
6. $Im(Z) = \frac{Z + \bar{Z}}{2i}$

<u>Argand Diagram</u>

There are two geometric representation of the Complex number $Z = x + iy$

I. As the point $p(x, y)$ in the xy-plane and C (complex plan) ω

II. As the vector $\overrightarrow{op}$ from origin to p

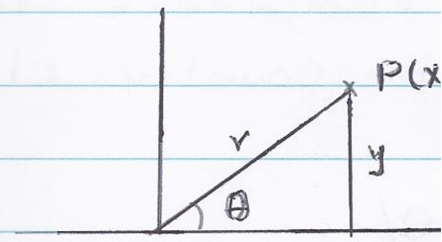


x – axis is real axis
y – axis is imaginary a

Vectorical representation of a complex number in the complex plan.

The polar form

Let $Z = p(x,y)$ be a point in the complex pl
then from the figure we have



$\because \cos\theta = \frac{x}{r}$

$\Rightarrow x = r\cos\theta$

$\because \sin\theta = \frac{y}{r}$

$\Rightarrow y = r\sin\theta$ s.t $r = |Z|$

$\because Z = x + iy$ $\therefore Z = r\cos\theta + ir\sin\theta$

wher r is called (modulus of Z) is the len
of vector $\overrightarrow{op}$ from origon to $p(x,y)$

$$\therefore r = |Z| = |x + iy| = \sqrt{x^2 + y^2}$$

polar angle θ is the argument of Z deno
by $\arg Z = \theta$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\arg(z) = \theta + 2k\pi \quad \text{where } k = 0, 1, 2, \ldots$$

and since $\sin\theta$, $\cos\theta$ are periodic with
period 2π
and θ is called argument principle and

denoted by $\text{Arg } Z = \theta$

The properties of $|Z|$

$\forall Z_1, Z_2, \dots, Z_n \in \mathbb{C}$ we have

1- $|Z_1 . Z_2| = |Z_1| . |Z_2|$ and
$|Z_1 . Z_2, \dots, Z_n| = |Z_1| . |Z_2| \dots |Z_n|$

2- $\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|}$

3- $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$ and
$|Z_1 + Z_2 + Z_3 + \dots + Z_n| \leq |Z_1| + \dots + |Z_n|$

4- $|Z_1 - Z_2| \geq |Z_1| - |Z_2|$

5- $|Z| \geq |Re(Z)| \geq Re(Z)$

6- $\big| |Z_1| - |Z_2| \big| \leq |Z_1 + Z_2|$

7- $|Z|^2 = (Re(Z))^2 + (Im(Z))^2$

Proof (3) $|Z_1 + Z_2| \geq |Z_1| - |Z_2|$

proof:-

for any two complex number's Z_1, Z_2 we can establish

$$|Z_1 + Z_2|^2 = (Z_1 + Z_2)\overline{(Z_1 + Z_2)}$$
$$= (Z_1 + Z_2)(\overline{Z_1} + \overline{Z_2})$$
$$= Z_1 . \overline{Z_1} + Z_2 . \overline{Z_1} + Z_1 . \overline{Z_2} + Z_2 . \overline{Z_2}$$
$$= |Z_1|^2 + |Z_2|^2 + 2Re(Z_1 . \overline{Z_2})$$

by observing that $Re(Z_1 . \overline{Z_2}) \leq |Z_1 . \overline{Z_2}|$,
we have

$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1 . \bar{z}_2|$

by take the positive sequare root on both si
, and obtain

$|z_1 + z_2| \leq |z_1| + |z_2|$

proof (6) $||z_1| - |z_2|| \leq |z_1 + z_2|$

$|z_1| = |(z_1 + z_2) + (-z_2)|$

$\leq |z_1 + z_2| + |z_2|$

$|z_1| - |z_2| \leq |z_1 + z_2|$

$|z_2| = |z_2 + z_1 + (-z_1)|$

$\leq |z_1 + z_2| + |z_1|$

$\therefore |z_2| - |z_2| \leq |z_1 + z_2|$

$\therefore ||z_1| - |z_2|| \leq |z_1 + z_2|$

Exaple write the polar form $Z = 2 - 2i$

solution

$x = 2$, $y = -2$ $\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{8}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\arg(Z) = \theta \mp 2k\pi$ s.t $k = 0, 1, 2, \ldots$

$Z = r(\cos\theta + i\sin\theta)$

$= \sqrt{8}\left(\cos\left(-\frac{\pi}{4} \mp 2k\pi\right) + i\sin\left(-\frac{\pi}{4} \mp 2k\pi\right)\right)$

H.W put $Z = 1 + \sqrt{3}\,i$ in polar form

H.W

① $|1 + Z\bar{w}|^2 + |Z - w|^2 = (1 + |Z|^2) \cdot (1 + |w|^2)$

② $|Z - w|^2 - |Z + \bar{w}| = -4 \operatorname{Re} Z \cdot \operatorname{Re} w$

③ Show that $\sqrt{2}\,|Z| \geq (\operatorname{Re}(Z) + \operatorname{Im}(Z))$

④ Prove that $\big||Z_1| - |Z_2|\big| \leq |Z_1 - Z_2| \leq |Z_1| + |Z_2|$

⑤ Z either real or imag. iff $(\bar{Z})^2 = Z^2$

De' Moivre's theorem

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$

$Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then $Z_1 . Z_2 = r_1(\cos\theta_1 + i\sin\theta_1) . r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore Z_1 . Z_2 = r_1 . r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

where we can generalli:zed the lest as followes.

if $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ ... $Z_n = r_n(\cos\theta_n + i\sin\theta_n)$

then $Z_1 . Z_2 ... Z_n = r_1 . r_2 ... r_n [\cos(\theta_1 + \theta_2 + ... +$
$+ i[\sin(\theta_1 + \theta_2 + ... + \theta_n]$

$\therefore Z_1 = Z_2 = ... Z_n = Z$

$\therefore Z^n = r^n[\cos n\theta + i\sin n\theta]$ ... (1)

$\because Z = r(\cos\theta + i\sin\theta)$

$\therefore Z^n = r^n(\cos\theta + i\sin\theta)^n$ ... (2)

for equ. (1) and (2) find

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ ... (

This equation called De' Moivre's theorem

De' Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

proof:-

by using Mathematical induction.

The theorem is true for $n=1$ that is

$$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$$

Now suppose the theorem is true for $n=k$

to prove that inquality is true for $n=k+1$

$$(\cos\theta + \sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$$

$$= (\cos k\theta . \cos\theta - \sin k\theta \sin\theta) +$$

$$i(\cos k\theta . \sin\theta - \sin k\theta \cos\theta)$$

$$= (\cos(k\theta + \theta) + i\sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

Roots of the Complex numbers:-

A number w is called the n-th root of the Complex number Z if

$$W^n = Z \text{ or } Z^{\frac{1}{n}} = W$$

Thus $W = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}}$

$$= r^{\frac{1}{n}} [\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right.$$

where $k = 0, 1, 2, \cdots$

Example (1) solve $Z^3 = 1$

Solution

know $n = 3$, $r = \sqrt{x^2+y^2} \Rightarrow r = \sqrt{1} = 1$

$\theta = \tan^{-1}\frac{y}{x}$ where $\sin\theta = \frac{y}{r} = 0$, $\cos\theta = \frac{x}{r} = 1$

$\therefore \theta = 0$

$3\theta = \theta + 2k\pi$ since $(\arg(Z_1 \cdot Z_2 \cdots Z_n) = \arg Z_1 +$
$\therefore \theta = \frac{2}{3}k\pi$ $\arg Z_2 + \cdots + \arg Z_n$
$\therefore \arg Z^n = n \arg Z)$

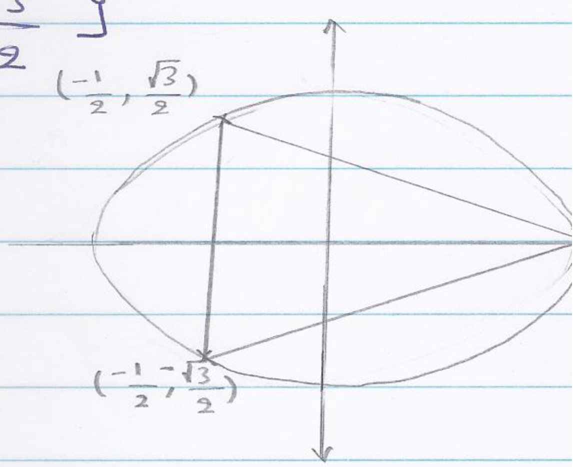
$W_k = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right]$

$W_0 = \cos 0 + i\sin 0 = 1$ where $k = 0$

$W_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\sqrt{3}/2$

$W_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -\frac{1}{2} - i\sqrt{3}/2$

$\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$



The equation $Z^3 = 1$ have three different roots and this roots represent Geometricaly Trian heads drown inside Circle Centre origon p and radius is 1

Example (2) solve $Z^3 = -1$

Solution

know $n = 3$ $\quad r = \sqrt{x^2 + y^2} \Rightarrow r = \sqrt{1} = 1$

$\theta = \tan^{-1} \frac{y}{x}$ where $\sin\theta = \frac{y}{r} = 0, \; \cos\theta = \frac{x}{r} = -$

$\therefore \theta = \pi$

$\therefore 3\theta = \theta + 2k\pi$

$\therefore \theta = \frac{\pi + 2k\pi}{3}$ $\quad$ where $k = 0, 1, 2, \ldots$

$$W_k = r^{\frac{1}{n}} \left[\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right]$$

$$W_0 = r^{\frac{1}{3}} \left[\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right] = \frac{1}{2} + i \frac{\sqrt{3}}{2}$$

$W_1 =$

$W_2 =$

Example (3) Solve $Z^5 = 1$ H.w

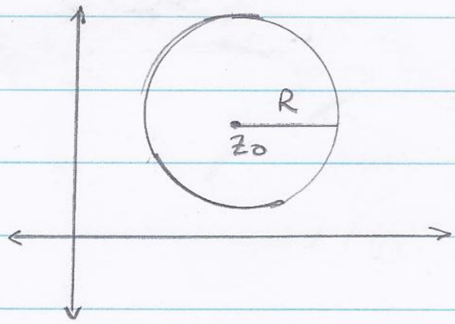
solution

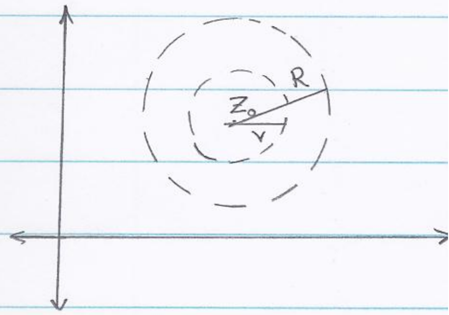
Regions in the Complex plane

Before making any definitions, let us consider a
examples of sets, which we need:

Example (1)

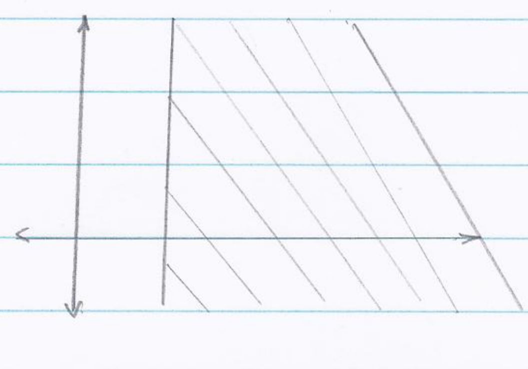
Suppose that $z_0 \in \mathbb{C}$, r, R and $0 < r < R$. The se
set $\{z \in \mathbb{C} : |z - z_0| < R\}$ represents a disc,
with center z_0, and radius R
the set $\{z \in \mathbb{C} : r < |z - z_0| < R\}$ represents a
annulus with radius r, and outer radius R

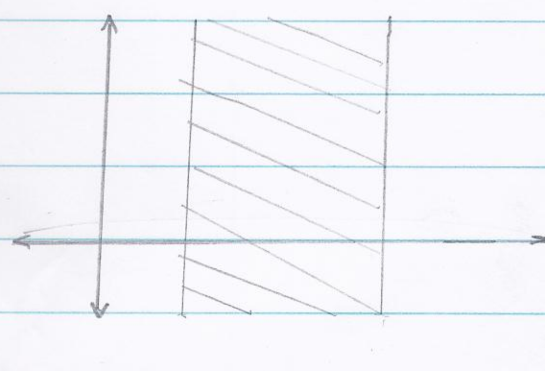




Example (2)

Suppose that $A, B \in \mathbb{R}$ and $A < B$, the set
$\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{R}$ and $x > A\}$ represent
a half-plane, and
the set $\{z = x + iy : x, y \in \mathbb{R}$ and $A < x < B\}$ repre
a strip





Definition: A neighbourhood of radius r, of the point Z_0 is the collection of all the points inside A circle of radius r, centered at Z_0, These are the points satisfying $|Z - Z_0| < r$

Definition A deleted neighbourhood is a nbd consisting of all points Z in an ϵ-neighbourhood of Z_0 except for the point Z_0 itself.

$$0 < |Z - Z_0| < \epsilon$$

Definition suppose that S is a point set in $\mathbb{C}$. A point $Z_0 \in S$ is said to be interior point of S if there exists an ϵ-neighbourhood of Z_0 which is contained in S. The set S is said to be open if every point of S is an interior point of

Example

(1) $\{Z \in \mathbb{C} ; |Z - Z_0| < R\}$ open.

(2) $\{Z \in \mathbb{C} ; r < |Z - Z_0| < R\}$ open.

(3) $\{Z = r(\cos\theta + i\sin\theta) \in \mathbb{C}, r, \theta$ and $r > 0$, $\alpha < \theta < \beta\}$ a sector is open.

(4) $\{Z \in \mathbb{C}, 0 < |Z - Z_0| < R\}$ is open.

(5) $\{Z \in \mathbb{C}, |Z - Z_0| \leq R\}$ is not open.

(6) the empty set $\emptyset$ is open.